Semantical Considerations on Description Logics of MKNF

Anees Mehdi

anees.mehdi@kit.edu
Institute AIFB,
Karlsruhe Institute of Technology,
Germany

Abstract

Description logics of minimal knowledge and negation as failure (MKNF-DLs) are formalisms which augment description logics (DLs) with the modal operators $\mathbf{K}$ representing 'knowledge' and $\mathbf{A}$ representing 'default assumption'. Such hybrid formalisms are useful in characterizing many nonmonotonic features which can not captured in pure DLs. The traditional semantics employed for MKNF-DLs is based on the possible world approach where each world corresponds to a DL interpretation. Further, the semantics requires the interpretations to share a common domain and to interpret constants rigidly across the worlds. In this paper we argue that these restrictions lead to unintended effects when an expressive MKNF-DL like $\mathcal{SROIQK}_{\mathcal{NF}}$ is considered. We thus propose employing the extended semantics, introduced recently, for $\mathcal{SROIQK}_{\mathcal{NF}}$. We then provide a comparison between the traditional and the extended semantics including a comparison from first-order modal logic perspective. In addition, we present a methodology for performing reasoning tasks in $\mathcal{SROIQK}_{\mathcal{NF}}$.

Introduction

The origin of Description logics (DLs) dates back to the quest for semantics for knowledge representation systems like Frame Based Systems (Minsky 1974). DLs are decidable fragments of first-order logic (Baader et al. 2007) and thus the semantics employed for DLs is inherently based on open-world assumption. Such an assumption is very handing in modeling problem domains with incomplete knowledge such as Semantic Web (Hitzler, Krötzsch, and Rudolph 2009). However, several features of frame-based system are expressible only under the close world assumption. Besides, several problem domain requires some sort of non-monotonic reasoning. Consequently, several work has been done on extending DLs with non-standard features in order to capture defeasible inferencing (Donini, Nardi, and Rosati 2002)▷**check:more ref**◁. However in this work we mainly focus on MKNF-DLs: standard DLs augmented with modal operators $\mathbf{K}$ (representing knowledge) and $\mathbf{A}$ (representing default assumption).

While propositional logic extended by the modal operators has been widely studied and is well-understood, the introduction of the operators into first-order logic (as treated by Fitting and Mendelsohn 1998 and Braüner and Ghilardi 2006) brings about conceptual controversies concerning assumptions to be made about the domains of quantification, equality, (non-)rigidity of constants and the like. Early works on extending DLs with the operator $\mathbf{K}$ and/or the operator $\mathbf{A}$ include (Donini et al. 1992,Donini, Nardi, and Rosati 1995,Donini, Nardi, and Rosati 1997,Donini, Nardi, and Rosati 2002, etc.). The formalism presentd in (Donini, Nardi, and Rosati 2002), called $\mathcal{ALCK}_{\mathcal{NF}}$ extends the DL $\mathcal{ALC}$ (Schmidt-Schauß and Smolka 1991) with both aforementioned operators and thus is more expressive than the other approaches of extending a DL with the operator $\mathbf{K}$ only. Consequently many non-monotonic capabilities are acquired. The semantics defined for these extensions are based on the possible-world approach, where each world corresponds to a DL interpretation wheres as the domain of quantification is fixed to a common countable infinite set and the constants are interpreted rigidly.

In this work, we first notice that when extending expressive DLs like $\mathcal{SROIQ}$ (Horrocks, Kutz, and Sattler 2006), the traditional semantics can not be employed. In fact, $\mathcal{SROIQ}$ is expressive enough to formalized axioms which only allow models with finite domain. We thus take the approach presented in (Mehdi and Rudolph 2011) for defining the semantics of expressive MKNF-DL like $\mathcal{SROIQ}$ extended with $\mathbf{K}$ and $\mathbf{A}$. We than provide a comparison between our semantics with the traditional one (while restricting the underlying formalism) from first-order modal logic perspective (Fitting and Mendelsohn 1998). We then extended the traditional reasoning approach such the one presented in (Donini, Nardi, and Rosati 2002) to our approach. To this end we provide an algorithm for deciding the basic tasks like entailment and knowledge base consistency.

We omit most of the numerous and lengthy proofs from the paper and refer the interested reader to the accompanying technical report (Mehdi 2013).

Preliminaries

Our results are independent of the fragment of first-order logic under consideration, although we focus on DL $\mathcal{SROIQ}$(Horrocks, Kutz, and Sattler 2006) only. In this work, we mainly use the description logic $\mathcal{SROIQ}$. Though our results are application

Description Logic $\mathcal{SROIQ}$

The DL $\mathcal{SROIQ}$ (Horrocks, Kutz, and Sattler 2006) provides the foundations for OWL 2 DL, the most comprehensive version of OWL that still allows for automated reasoning (Hitzler, Krötzsch, and Rudolph 2009). As for the signature of $\mathcal{SROIQ}$, let N_I, N_C, and N_R be finite, disjoint sets called *individual names*, *concept names* and *role names* respectively, with N_R partitioned into *simple* and *non-simple* roles. These atomic entities can be used to form complex ones as displayed in Table 1.

Name	Syntax	Semantics
inverse role	R^-	$\{(x,y) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (y,x) \in R^\mathcal{I}\}$
universal role	U	$\Delta^\mathcal{I} \times \Delta^\mathcal{I}$
top	$\top$	$\Delta^\mathcal{I}$
bottom	$\bot$	$\emptyset$
negation	$\neg C$	$\Delta^\mathcal{I} \setminus C^\mathcal{I}$
conjunction	$C \sqcap D$	$C^\mathcal{I} \cap D^\mathcal{I}$
disjunction	$C \sqcup D$	$C^\mathcal{I} \cup D^\mathcal{I}$
nominals	$\{a_1, \ldots, a_n\}$	$\{a_1^\mathcal{I}, \ldots, a_n^\mathcal{I}\}$
univ. restriction	$\forall R.C$	$\{x \mid \forall y.(x,y) \in R^\mathcal{I} \to y \in C^\mathcal{I}\}$
exist. restriction	$\exists R.C$	$\{x \mid \exists y.(x,y) \in R^\mathcal{I} \wedge y \in C^\mathcal{I}\}$
Self concept	$\exists S.\mathsf{Self}$	$\{x \mid (x,x) \in S^\mathcal{I}\}$
qualified number	$\leqslant n\, S.C$	$\{x \mid \#\{y \in C^\mathcal{I} \mid (x,y) \in S^\mathcal{I}\} \leq n\}$
restriction	$\geqslant n\, S.C$	$\{x \mid \#\{y \in C^\mathcal{I} \mid (x,y) \in S^\mathcal{I}\} \geq n\}$

Table 1: Syntax and semantics of role and concept constructors in $\mathcal{SROIQ}$. Thereby a denotes an individual name, R an arbitrary role name and S a simple role name. C and D denote concept expressions.

Axiom α	$\mathcal{I} \models \alpha$, if	
$R_1 \circ \cdots \circ R_n \sqsubseteq R$	$R_1^\mathcal{I} \circ \cdots \circ R_n^\mathcal{I} \subseteq R^\mathcal{I}$	RBox $\mathcal{R}$
$\mathsf{Dis}(S,T)$	$S^\mathcal{I} \cap T^\mathcal{I} = \emptyset$	
$C \sqsubseteq D$	$C^\mathcal{I} \subseteq D^\mathcal{I}$	TBox $\mathcal{T}$
$C(a)$	$a^\mathcal{I} \in C^\mathcal{I}$	ABox $\mathcal{A}$
$R(a,b)$	$(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$	
$a \doteq b$	$a^\mathcal{I} = a^\mathcal{I}$	
$a \not\doteq b$	$a^\mathcal{I} \neq b^\mathcal{I}$	

Table 2: Syntax and semantics of $\mathcal{SROIQ}$ axioms

A *$\mathcal{SROIQ}$ knowledge base (or $\mathcal{SROIQ}$ KB in short)* is a tuple $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ where $\mathcal{T}$ is a $\mathcal{SROIQ}$ TBox, $\mathcal{R}$ is a $\mathcal{SROIQ}$ role hierarchy[1] and $\mathcal{A}$ is a $\mathcal{SROIQ}$ ABox. Table 2 presents the respective axiom types.

The semantics of $\mathcal{SROIQ}$ is defined via interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ composed of a non-empty set $\Delta^\mathcal{I}$ called the *domain of* $\mathcal{I}$ and a function $\cdot^\mathcal{I}$ mapping individual names to elements of $\Delta^\mathcal{I}$, concept names to subsets of $\Delta^\mathcal{I}$ and role names to subsets of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. This mapping is extended to complex role and concept expressions as in Table 1 and finally used to define satisfaction of axioms (see Table 2). We say that $\mathcal{I}$ satisfies a knowledge base $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ (or $\mathcal{I}$ is a model of Σ, written: $\mathcal{I} \models \Sigma$) if it satisfies all axioms of $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{A}$. We say that a knowledge base Σ *entails* an axiom α (written $\Sigma \models \alpha$) if all models of Σ are models of α.

[1]We assume the usual regularity assumption for $\mathcal{SROIQ}$, but omit it for space reasons.

Description Logics of MKNF

In (Donini, Nardi, and Rosati 2002), the formalism $\mathcal{ALCK_{NF}}$ is presented which extends the description logic $\mathcal{ALC}$[2] with two modalities namely, $\mathbf{K}$ (minimal knowledge) and $\mathbf{A}$ (negation as failure). Syntactically, $\mathcal{ALCK_{NF}}$ can be taken as a fragment of the (first-order) logic of minimal knowledge and negation as failure (MKNF (Lifschitz 1991)). Though for semantics, we will see in the following that certain restrictions are imposed.

We first present the syntax of $\mathcal{ALCK_{NF}}$. *A* $\mathcal{ALCK_{NF}}$ role R is given by the following grammar

$$R ::= S|\mathbf{K}R|\mathbf{K}R$$

where S is a role name. Similarly, for an $\mathcal{ALCK_{NF}}$ role R and a concept name A, *an* $\mathcal{ALCK_{NF}}$ *concept* is given by the following grammar

$$C ::= \top|\bot|A|C \sqcap C|C \sqcup C|\neg C|\exists R.C|\exists R.C| \\ \forall R.C|\mathbf{K}C|\mathbf{A}C$$

The notions of axioms and KBs are defined in the obvious way. By definition every $\mathcal{ALC}$ concept/role/axiom/KB is an $\mathcal{ALCK_{NF}}$ concept/role/axiom/KB. Note that any DL can be extended with the operators $\mathbf{K}$ and $\mathbf{A}$. In sequel by MKNF-DL we mean a DL extended with $\mathbf{K}$ and $\mathbf{A}$.

Classical Semantics The semantics for $\mathcal{ALCK_{NF}}$ is defined in terms of MKNF structures: *sets of* $\mathcal{ALC}$ *interpretations.*[3] Under the classical semantics, the following assumptions are made:

- all $\mathcal{ALC}$ interpretations under consideration share a common domain which is a countably infinite set containing all the individual names, and
- the interpretation of any individual names remains the same across different interpretations.

An $\mathcal{ALCK_{NF}}$ *interpretation* is a triple $(\mathcal{I}, \mathcal{M}, \mathcal{N})$ where $\mathcal{I}$ is an $\mathcal{ALC}$ interpretation, and $\mathcal{M}$ and $\mathcal{N}$ are sets of $\mathcal{ALC}$ interpretations. All these interpretations have a countably infinite set Δ as their domain such that $N_I \subset \Delta$. The interpretation of an $\mathcal{ALCK_{NF}}$ concept C in an epistemic interpretation $(\mathcal{I}, \mathcal{M}, \mathcal{N})$, denoted by $C^{\mathcal{I},\mathcal{M},\mathcal{N}}$, can be defined inductively as given in Table 3

The satisfaction of axiom, TBoxes, ABoxes, RBoxes and knowledge bases, can be defined in a straight forwardly.

[2]$\mathcal{ALC}$ is the least expressive boolean complete DL that allows for universal and existential restriction constructs.

[3]In such structures, each interpretation corresponds to a possible world(Kripke 1971). Since the accessibility relation is assumed to be total, it is not explicitly given.

Role S	$S^{\mathcal{I},\mathcal{M},\mathcal{N}}$
P	$P^{\mathcal{I}}$ for an $\mathcal{ALC}$ role P
$\mathbf{K}P$	$\bigcap_{\mathcal{J}\in\mathcal{M}}(P)^{\mathcal{J},\mathcal{M},\mathcal{N}}$
$\mathbf{A}P$	$\bigcap_{\mathcal{J}\in\mathcal{N}}(P)^{\mathcal{J},\mathcal{M},\mathcal{N}}$
Concept X	$X^{\mathcal{I},\mathcal{M},\mathcal{N}}$
$\top$	Δ
$\bot$	$\emptyset$
C	$C^{\mathcal{I}}$ for an $\mathcal{ALC}$ concept C
$\neg C$	$\Delta\setminus(C)^{\mathcal{I},\mathcal{M},\mathcal{N}}$
$C_1\sqcup C_2$	$C_1^{\mathcal{I},\mathcal{M},\mathcal{N}}\cup C_2^{\mathcal{I},\mathcal{M},\mathcal{N}}$
$C_1\sqcap C_2$	$C_1^{\mathcal{I},\mathcal{M},\mathcal{N}}\cap C_2^{\mathcal{I},\mathcal{M},\mathcal{N}}$
$\exists R.C$	$\{d\in\Delta\mid\exists d'$ with $(d,d')\in R^{\mathcal{I},\mathcal{M},\mathcal{N}}$ and $d'\in C^{\mathcal{I},\mathcal{M},\mathcal{N}}\}$
$\forall R.C$	$\{d\in\Delta\mid\forall d'$ with $(d,d')\in R^{\mathcal{I},\mathcal{M},\mathcal{N}}$ implies $d'\in C^{\mathcal{I},\mathcal{M},\mathcal{N}}\}$
$\mathbf{K}C$	$\bigcap_{\mathcal{J}\in\mathcal{M}}(C)^{\mathcal{J},\mathcal{M},\mathcal{N}}$
$\mathbf{A}C$	$\bigcap_{\mathcal{J}\in\mathcal{N}}(C)^{\mathcal{J},\mathcal{M},\mathcal{N}}$

Table 3: Semantics of role and concept constructors in $\mathcal{ALCK}_{\mathcal{NF}}$. Thereby P is a role name, R is an $\mathcal{ALCK}_{\mathcal{NF}}$ role and C_1, C_2 and C are $\mathcal{ALCK}_{\mathcal{NF}}$ concepts

Given an $\mathcal{ALCK}_{\mathcal{NF}}$ knowledge base $\mathcal{O}$, a set of $\mathcal{ALC}$ interpretations $\mathcal{M}$ is called *an $\mathcal{ALCK}_{\mathcal{NF}}$ model* (written $\mathcal{M}\models\mathcal{O}$) iff for every $\mathcal{I}\in\mathcal{M}$ we have that $(\mathcal{I},\mathcal{M},\mathcal{M})$ satisfies $\mathcal{O}$ and for any set of $\mathcal{ALC}$ interpretation $\mathcal{M}'$ with $\mathcal{M}\subset\mathcal{M}'$ there is some $\mathcal{I}'\in\mathcal{M}'$ such that $(\mathcal{I},\mathcal{M}',\mathcal{M})$ does not satisfy $\mathcal{O}$. We call an $\mathcal{ALCK}_{\mathcal{NF}}$ model of $\mathcal{O}$ just a model of $\mathcal{O}$ whenever its clear from the context. Note how non-monotonicity is acquired in the semantics by preferring maximal sets as the models. In this way, we introduce minimality for $\mathbf{K}$ and default assumption for $\mathbf{A}$.

Now the notion of entailment in $\mathcal{ALCK}_{\mathcal{NF}}$ is defined as follows. An $\mathcal{ALCK}_{\mathcal{NF}}$ knowledge base $\mathcal{O}$ *entails an $\mathcal{ALCK}_{\mathcal{NF}}$ axiom* α if and only if for every model $\mathcal{M}$ of $\mathcal{O}$ is such that $(\mathcal{I},\mathcal{M},\mathcal{M})$ satisfies α i.e., $(\mathcal{I},\mathcal{M},\mathcal{M})\models\alpha$. Unlike the standard entailment, we write $\mathcal{O}\mid\models\alpha$.

Note that by definition every standard DL knowledge base Σ is an MKNF-DL knowledge base with unique $\mathcal{SRIQK}_{\mathcal{NF}}\setminus U$ model $\mathcal{M}(\Sigma)$, which is the set of all standard models of Σ with a countably infinite domain Δ.

With the modal operators $\mathbf{K}$ and $\mathbf{A}$, $\mathcal{ALC}_{\mathcal{NF}}$ can be used to model several features of frame-based systems (Minsky 1974) like defaults, integrity constraints, concept/role closure (see Donini, Nardi, and Rosati 2002). For example, consider the wine ontology.[4]. To enforce a constraint that for any wine it should be known if it is a white wine or a red wine, the following axiom can be added to the knowledge base.

$$\mathbf{K}\mathit{Wine}\sqsubseteq\mathbf{A}\mathit{RedWine}\sqcup\mathbf{A}\mathit{WhiteWine}$$

As mentioned in (Donini, Nardi, and Rosati 2002) a similar axiom using only the $\mathbf{K}$ operators is incorrect formalization of the aforementioned constraint. Nevertheless, using $\mathbf{K}$ instead of $\mathbf{A}$ in axioms has no effect so far entailment (from a knowledge base) of such an axiom is in question.

[4]http://www.w3.org/TR/owl-guide/wine.rdf

Similar to $\mathcal{ALCK}_{\mathcal{NF}}$ we can extended $\mathcal{SROIQ}$ to obtained $\mathcal{SROIQK}_{\mathcal{NF}}$. However as shown in (Mehdi and Rudolph 2011), the semantics can not be employed as is. In $\mathcal{SROIQ}$ we can express axioms that allow for models with only finite domains. For example a KB containing the axioms $\top\sqsubseteq\{a,b,c\}$ or $\top\sqsubseteq\leq 3U.\top$ has models with at most 3 elements in their domains. Whereas in $\mathcal{ALCK}_{\mathcal{NF}}$ every interpretation considered has Δ (a countably infinite set) as its domain. To over come this problem, the notion of extended interpretation is presented in (Mehdi and Rudolph 2011) for $\mathcal{SROIQK}$ ($\mathcal{SROIQ}$ extended with $\mathbf{K}$). We adopt the same approach here for $\mathcal{SROIQK}_{\mathcal{NF}}$.

Extended Semantics

The extended semantics introduces an abstraction layer in the standard interpretation to obtain an extended interpretations. This layer assigns *abstract individual names* to domain elements. The names are from the set $N_I\cup\mathbb{N}$ and hence common to all interpretations, thus they can serve as the "common ground" for different interpretations with different domains. It is required that every domain element is associated with at least one abstract name, however, different names can denote the same domain element (thus allowing for the possibility of finite domains).

Definition 1. *An* extended $\mathcal{SROIQ}$-interpretation $\tilde{\mathcal{I}}$ *is a tuple* $(\Delta_{\tilde{\mathcal{I}}},\cdot^{\tilde{\mathcal{I}}},\varphi_{\tilde{\mathcal{I}}})$ *such that*

- $(\Delta_{\tilde{\mathcal{I}}},\cdot^{\tilde{\mathcal{I}}})$ *is a standard DL interpretation,*
- $\varphi_{\tilde{\mathcal{I}}}:N_I\cup\mathbb{N}\twoheadrightarrow\Delta^{\mathcal{I}}$ *is a surjective function from* $N_I\cup\mathbb{N}$ *onto* $\Delta^{\mathcal{I}}$*, such that for all* $a\in N_I$ *we have that* $\varphi_{\tilde{\mathcal{I}}}(a)=a^{\tilde{\mathcal{I}}}$.

We call $\varphi_{\tilde{\mathcal{I}}}$ *as the abstraction mapping of* $\tilde{\mathcal{I}}$*. This mapping returns the actual interpretation of an individual, given its (abstract) name, under the interpretation* $\tilde{\mathcal{I}}$*. We extend the definition of* $\varphi_{\tilde{\mathcal{I}}}$ *to subsets of* $N_I\cup\mathbb{N}$*. For a set* S, $\varphi_{\tilde{\mathcal{I}}}(S):=\{\varphi_{\tilde{\mathcal{I}}}(t)\mid t\in S\}$*. Similarly we extend* $\varphi_{\tilde{\mathcal{I}}}$ *to ordered pairs and set of ordered pairs on* $N_I\cup\mathbb{N}$ *as follows:*

- $\varphi_{\tilde{\mathcal{I}}}((s,t)):=(\varphi_{\tilde{\mathcal{I}}}(s),\varphi_{\tilde{\mathcal{I}}}(t))$ *for ordered pairs* $(s,t)\in(N_I\cup\mathbb{N})^2$.
- $\varphi_{\tilde{\mathcal{I}}}(T):=\{\varphi_{\tilde{\mathcal{I}}}((s,t))\mid(s,t)\in T\}$ *for sets* $T\subseteq(N_I\cup\mathbb{N})^2$.

We also define the inverse $\varphi_{\tilde{\mathcal{I}}}^{-1}$ *of the mapping* $\varphi_{\tilde{\mathcal{I}}}$ *for an extended interpretation* $\tilde{\mathcal{I}}$ *as follows:*

- $\varphi_{\tilde{\mathcal{I}}}^{-1}(x):=\{t\in N_I\cup\mathbb{N}\mid\varphi_{\tilde{\mathcal{I}}}(t)=x\}$ *for every* $x\in\Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}(E):=\bigcup_{x\in E}\varphi_{\tilde{\mathcal{I}}}^{-1}(x)$ *for* $E\subseteq\Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}((x,y)):=\{(s,t)\mid\varphi_{\tilde{\mathcal{I}}}((s,t))=(x,y)\}$ *for ordered pairs* $(x,y)\in\Delta^{\tilde{\mathcal{I}}}\times\Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}(H):=\bigcup_{(x,y)\in H}\varphi_{\tilde{\mathcal{I}}}^{-1}((x,y))$ *for any* $H\subseteq\Delta^{\tilde{\mathcal{I}}}\times\Delta^{\tilde{\mathcal{I}}}$.

. $\diamondsuit$

Based on these notions, an *extended $\mathcal{SROIQK}_{\mathcal{NF}}$ interpretation* for $\mathcal{SROIQK}$ is a tuple $(\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}})$, where $\tilde{\mathcal{I}}$ is an extended $\mathcal{SROIQ}$-interpretation, and $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$are sets of extended $\mathcal{SROIQ}$-interpretations. Similar to $\mathcal{ALCK}_{\mathcal{NF}}$ interpretations, one can define an extended interpretation function $\cdot^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}}$ as in Table 3 except:

$$(\mathbf{K}C)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}} = \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}}\in\tilde{\mathcal{M}}} \varphi_{\tilde{\mathcal{J}}}^{-1}\Big(C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}}\Big)$$

$$(\mathbf{K}R)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}} = \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}}\in\tilde{\mathcal{M}}} \varphi_{\tilde{\mathcal{J}}}^{-1}\Big(R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}}\Big)$$

$$(\mathbf{A}C)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}} = \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}}\in\tilde{\mathcal{N}}} \varphi_{\tilde{\mathcal{J}}}^{-1}\Big(C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}}\Big)$$

$$(\mathbf{A}R)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}} = \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}}\in\tilde{\mathcal{N}}} \varphi_{\tilde{\mathcal{J}}}^{-1}\Big(R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{N}}}\Big)$$

All the other notions are similar to the traditional semantics. We next compare the different semantics introduced in this section. Again similar to the traditional semantics , note that a $\mathcal{SROIQ}$ KB Σ has a unique extended $\mathcal{SROIQK}_{\mathcal{NF}}$ model. To define this model, by $\mathcal{E}(\mathcal{I})$ we mean the set of all extended interpretation obtained from the standard interpretation $\mathcal{I}$ with all possible abstraction mappings $\varphi_{\tilde{\mathcal{I}}}$. Similarly, for a set $\mathcal{I}$ of interpretation we define

$$\mathcal{E}(\mathcal{M}) := \bigcup_{\mathcal{I}\in\mathcal{M}} \mathcal{E}(\mathcal{I})$$

Now as Σ is $\mathbf{K}$ and $\mathbf{A}$ free KB and as in an extended interpretation $\tilde{\mathcal{I}}$, the abstraction mapping $\varphi_{\tilde{\mathcal{I}}}$ plays no role in the interpretation of $\mathbf{K}$ and $\mathbf{A}$ free axioms, we immediately get the set $\tilde{\mathcal{M}}(\Sigma) := \mathcal{E}(\mathcal{M}(\Sigma))$ as the unique extended model of Σ.

As of the reasoning tasks, by *consistency problem* of a given $\mathcal{SROIQK}_{\mathcal{NF}}$ KB Σ, we mean the problem of determining if Σ exhibits an extended $\mathcal{SROIQK}_{\mathcal{NF}}$ model. Similarly by *entailment problem* of an axiom α we mean the problem of deciding if $\Sigma \mid\!\models_{\mathrm{e}} \alpha$. It is easy to see that entailment problem can be reduced to consistency problem similar to the standard DLs (Baader et al. 2007).

The Traditional Semantics and $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$

As mentioned earlier, the reason of in-applicability of the classical semantics for expressive MKNF-DL is that there are MKNF-DL knowledge bases with models that have finite domains only. However, we next show that considering models with infinite domain suffices when we consider any MKNF-DL upto $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$[5]. The idea is that for any standard DL interpretation $\mathcal{I}$ with finite domain there is an interpretation with infinite domain that behaves exactly like $\mathcal{I}$ on the satisfaction of axioms provided the DL is restricted to $\mathcal{SRIQK}\backslash U$. This allows us to totally discard interpretation with finite domains from consideration. This is the reason why it suffices to define $\mathcal{ALCK}_{\mathcal{NF}}$ model as set of interpretations with a countably infinite common domain. To this end, by *lifting of an interpretation $\mathcal{I}$ to ω*, we mean the interpretation $\mathcal{I}_\omega$ obtained as follows:

- $\Delta^{\mathcal{I}_\omega} := \Delta^{\mathcal{I}} \times \mathbb{N}$,
- $a^{\mathcal{I}_\omega} := \langle a^{\mathcal{I}}, 0\rangle$ for every $a \in N_I$,
- $A^{\mathcal{I}_\omega} := \{\langle x, i\rangle \mid x \in A^{\mathcal{I}} \text{ and } i \in \mathbb{N}\}$ for each concept name $A \in N_C$,
- $r^{\mathcal{I}_\omega} := \{(\langle x, i\rangle, \langle x', i\rangle) \mid (x, x') \in r^{\mathcal{I}} \text{ and } i \in \mathbb{N}\}$ for every role name $r \in N_R$.

By structural induction, it is easy to show that for any interpretation $\mathcal{I}$ we have

$$\langle x, i\rangle \in C^{\mathcal{I}_\omega} \text{ iff } x \in C^{\mathcal{I}} \qquad (1)$$

for any $x \in \Delta^{\mathcal{I}}$ and $\mathcal{SRIQK}\backslash U$ concept C. Consequently we get the following result.

Lemma 2. *Let Σ be a $\mathcal{SRIQ}$ knowledge base. For any interpretation $\mathcal{I}$ we have that*

$$\mathcal{I} \models \Sigma \textit{ if and only if } \mathcal{I}_\omega \models \Sigma.$$

Proof. First we note that it follows immediately from the definition of $\mathcal{I}_\omega$ that for any $\mathcal{SRIQ}$-role $R \in \mathbf{R}$ and $(\langle x, i\rangle, \langle y, i'\rangle) \in \Delta^{\mathcal{I}_\omega}$ for $i, i' \in \mathbb{N}$ we have that $(\langle x, i\rangle, \langle y, i'\rangle) \in R^{\mathcal{I}_\omega}$ if an only if $(x, y) \in R^{\mathcal{I}}$ and $i = i'$ for an interpretation $\mathcal{I}$. Now for any RIA $R_1 \circ \ldots R_n \sqsubseteq R$ we have that:
$\mathcal{I} \models R_1 \circ \ldots R_n \sqsubseteq R$
$\Leftrightarrow \mathcal{I} \models R_1^{\mathcal{I}} \circ \ldots R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$, whenever $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \le i \le n$ then $(x_0, x_n) \in R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$ and any $j \in \mathbb{N}$, whenever $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$ for $1 \le i \le n$ then $(\langle x_0, j\rangle, \langle x_n, j\rangle) \in R^{\mathcal{I}_\omega}$
$\Leftrightarrow \mathcal{I}_\omega \models R_1 \circ \ldots R_n \sqsubseteq R$.
The second last equivalence holds as $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \le i \le n$ and any non-negative integer j implies that $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$. Similary $(\langle x_{i-1}, j_{i-1}\rangle, \langle x_i, j_i\rangle) \in R^{\mathcal{I}_\omega}$ for $1 \le i \le n$ implies that $(x_{i-1}, x_i) \in R^{\mathcal{I}}$ and that all j_i, s are equal. And the same holds for the role R.

Similary, for any role characteristic $\mathsf{Ref}(R)$, we have that:
$\mathcal{I} \models \mathsf{Ref}(R)$
$\Leftrightarrow (x, x) \in R^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
$\Leftrightarrow (\langle x, j\rangle, \langle x, j\rangle) \in R^{\mathcal{I}_\omega}$ for any $j \in \mathbb{N}$ and $x \in \Delta^{\mathcal{I}}$
$\Leftrightarrow (\langle x, j\rangle, \langle x, j\rangle) \in R^{\mathcal{I}_\omega}$ for any $\langle x, j\rangle \in \Delta^{\mathcal{I}_\omega}$ as $\Delta^{\mathcal{I}_\omega} = \Delta^{\mathcal{I}} \times \mathbb{N}$
$\Leftrightarrow \mathcal{I}_\omega \models \mathsf{Ref}(R)$.
In the same way, we can prove for any of the rest of the role characteristics that whenever $\mathcal{I}$ models it so does $\mathcal{I}_\omega$. Consequently we have that for any role hierarchy $\mathcal{R}, \mathcal{I} \models \mathcal{R}$ if and only if $\mathcal{I}_\omega \models \mathcal{R}$.

Now for any GCI $C \sqsubseteq D$ and for any interpretation $\mathcal{I}$, by (1) we get $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ if and only if $C^{\mathcal{I}_\omega} \subseteq D^{\mathcal{I}_\omega}$. Further for any TBox $\mathcal{T}, \mathcal{I} \models \mathcal{T}$ if and only if $\mathcal{I}_\omega \models \mathcal{T}$.

Finally for an ABox $\mathcal{A}$ we show that for each assertion in $\alpha \in \mathcal{A}, \mathcal{I} \models \alpha$ if and only if $\mathcal{I}_\omega \models \alpha$.

- α is of the form $C(a)$: Now for an interpretation $\mathcal{I}$ it follows from the definition of $\mathcal{I}_\omega$ that $a^{\mathcal{I}_\omega} = (a^{\mathcal{I}}, 0)$.

[5] $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$ is $\mathcal{SROIQK}_{\mathcal{NF}}$ without nominals and the universal role.

As we have already shown that $a^{\mathcal{I}} \in C^{\mathcal{I}}$ if and only if $(a^{\mathcal{I}}, i) \in C^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$. Hence we get that $a^{\mathcal{I}} \in C^{\mathcal{I}}$ if and only if $(a^{\mathcal{I}}, 0) \in C^{\mathcal{I}_\omega}$.

- Analogously we can show an interpretation $\mathcal{I}$ satisfies an assertion if and only if $\mathcal{I}_\omega$ does so.

As a consequence of the above lemma, the restriction to interpretations with infinite domains suffices for the semantics for $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$ as required by the common domain assumption. Further, since we can define a one-to-one mapping between $\Delta^{\mathcal{I}}$ and Δ (as defined in $\mathcal{ALCK}_{\mathcal{NF}}$ interpretation), it suffices to consider interpretations only with Δ as their models.

Semantic Comparison

As both the traditional semantics and the extended semantics are employable for $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$, we now study the relationship between the entailment relation in both the semantics. For this, let $\models_{\approx}$ and $\models_{\simeq}$ be two entailment relations defined for a logic $\mathcal{L}$. Then $\models_{\approx}$ and $\models_{\simeq}$ are compatible in $\mathcal{L}$ if we have

$$\Sigma \models_{\simeq} \alpha \text{ iff } \Sigma \models_{\approx} \alpha$$

for every $\mathcal{L}$ KB Σ and $\mathcal{L}$ axiom α. We now show that $|\!\!\models$ and $|\!\!\models_{e}$ are compatible for $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$. To this end we present the following definition.

Let $\mathcal{M}$ be a set of standard DL interpretations and $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ be an interpretation in $\mathcal{M}$. *An extended interpretation $\tilde{\mathcal{I}}$ based on $\mathcal{I}$* is defined as follows:

- $\Delta^{\tilde{\mathcal{I}}} = \Delta = N_I \cup \mathbb{N}$,
- $a^{\tilde{\mathcal{I}}} = a^{\mathcal{I}} = a$ for $a \in N_I$,
- $A^{\tilde{\mathcal{I}}} = A^{\mathcal{I}}$ for concept name A,
- $R^{\tilde{\mathcal{I}}} = R^{\mathcal{I}}$ for a role name R,
- $\varphi_{\tilde{\mathcal{I}}}$ is an identity function on $N_I \cup \mathbb{N}$ i.e., $\varphi_{\tilde{\mathcal{I}}}(x) = x$ for each $x \in N_I \cup \mathbb{N}$.

Given a set of interpretation $\mathcal{M}$, the set of extended interpretations based on $\mathcal{M}$ is denoted by $\tilde{\mathcal{M}}$ as is given as

$$\tilde{\mathcal{M}} = \{\tilde{\mathcal{I}} \mid \tilde{\mathcal{I}} \text{ is based on some } \mathcal{I} \in \mathcal{M}\}$$

To this end we observe he following relationship.

Lemma 3. *Let $\mathcal{M}$ be a set of interpretations. Then for a given $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$ role R and concept C, we have*

- $(x, y) \in R^{(\mathcal{I},\mathcal{M},\mathcal{M})}$ *if and only if* $(x, y) \in R^{(\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{M}})}$
- $x \in C^{(\mathcal{I},\mathcal{M},\mathcal{M})}$ *if and only if* $x \in C^{(\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{M}})}$

for arbitrary $x, y \in N_I \cup \mathbb{N}$.

The proof is simply by induction on the structure of R and C (c.f. (Mehdi 2013)).

As a consequence of the above lemma we get the following results.

Corollary 4. *Let α be an MKNF-DL axiom and M be a set of interpretations. Then we have*

$$M \models \alpha \textit{ if and only if } \tilde{\mathcal{M}} \models \alpha$$

where $\tilde{\mathcal{M}}$ is the set based on M.

Specifically, for each model $\mathcal{M}$ of Σ, we have that $\tilde{\mathcal{M}} \models \Sigma$ where $\tilde{\mathcal{M}}$ is the set of extended interpretations based on $\mathcal{M}$.

Now similar properties can be proved for set of extended interpretations. Let $\tilde{\mathcal{I}} = (\Delta^{\tilde{\mathcal{I}}}, \cdot^{\tilde{\mathcal{I}}}, \varphi_{\tilde{\mathcal{I}}})$ be an extended interpretation. Then *an interpretation $\mathcal{I}$ based on $\tilde{\mathcal{I}}$* is defined as follows:

- $\Delta^{\mathcal{I}} = N_I \cup \mathbb{N}$
- $a^{\mathcal{I}} = a$ for $a \in N_I$
- $A^{\mathcal{I}} = \varphi_{\tilde{\mathcal{I}}}^{-1}(A^{\tilde{\mathcal{I}}})$ for concept name A
- $R^{\mathcal{I}} = \varphi_{\tilde{\mathcal{I}}}^{-1}(R^{\tilde{\mathcal{I}}})$ for role name R

For a set $\tilde{\mathcal{M}}$ of extended interpretations the set

$$M = \{\mathcal{I} \mid \mathcal{I} \text{ is based on } \tilde{\mathcal{I}} \in \tilde{\mathcal{M}}\}$$

is called as the set based on $\tilde{\mathcal{I}}$.

In the following we restrict the extended interpretations $\tilde{\mathcal{I}}$ such that $\Delta^{\tilde{\mathcal{I}}}$ to a countably infinite set and $\varphi_{\tilde{\mathcal{I}}}$ to a one-to-one mapping. We thus observe the following.

Lemma 5. *Let $\tilde{\mathcal{M}}$ be a set of extended interpretations such that each ext. interpretation $\tilde{\mathcal{I}}$ in $\tilde{\mathcal{M}}$ is such that $\Delta^{\tilde{\mathcal{I}}}$ is countably infinite and $\varphi_{\tilde{\mathcal{I}}}$ is just a one-to-one mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{I}}}$. Further let $\mathcal{M}$ be the set of interpretations based on $\tilde{\mathcal{M}}$ and let R be an MKNF-DL role and C be an MKNF-DL concept. Then for any $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$ and $x, y \in \Delta^{\tilde{\mathcal{I}}}$*

- *$(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{M}}}$ if and only if there are $u, v \in N_I \cup \mathbb{N}$ with $\varphi_{\tilde{\mathcal{I}}}^{-1}(x) = u$ and $\varphi_{\tilde{\mathcal{I}}}^{-1}(y) = v$ such that $(u, v) \in R^{\mathcal{M},\mathcal{M},\mathcal{M}}$, where $\mathcal{I}$ is the interpretation based on $\tilde{\mathcal{I}}$,*
- *similarly, $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{M}}}$ if and only if there is a $u \in N_I \cup \mathbb{N}$ with $\varphi_{\tilde{\mathcal{I}}}^{-1}(x) = u$such that $u \in R^{\mathcal{M},\mathcal{M},\mathcal{M}}$, where $\mathcal{I}$ is the interpretation based on $\tilde{\mathcal{I}}$.*

As a consequence we get the following result.

Corollary 6. *Let α be an MKNF-DL axiom. Then for any set $\tilde{\mathcal{M}}$ of extended interpretations such that each ext. interpretation $\tilde{\mathcal{I}}$ in $\tilde{\mathcal{M}}$ is such that $\Delta^{\tilde{\mathcal{I}}}$ is countably infinite and $\varphi_{\tilde{\mathcal{I}}}$ is just a one-to-one mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{I}}}$, we have*

$$\tilde{\mathcal{M}} \models \alpha \textit{ if and only if } \mathcal{M} \models \alpha$$

We now prove the compatibility between $|\!\!\models_{e}$ and $|\!\!\models$ in $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$.

Theorem 7. *The extended entailment relation $|\!\!\models_{e}$ and epistemic entailment relation $|\!\!\models$ are compatible in $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$.*

Proof We have to show that for any $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$ knowledge base Σ and axiom α we have

$$\Sigma \,|\!\!\models_{e} \alpha \text{ if and only if } \Sigma \,|\!\!\models \alpha$$

For the if direction, suppose that $\Sigma \,|\!\!\not\models \alpha$. This means there is an $\mathcal{SRIQK}_{\mathcal{NF}}\backslash U$ model $\mathcal{M}$ of Σ such that $\mathcal{M} \not\models \alpha$. Let $\tilde{\mathcal{M}}$ be the set of extended interpretations based on $\mathcal{M}$. By Corollary 4 we get that $\tilde{\mathcal{M}} \models \Sigma$ and $\tilde{\mathcal{M}} \not\models \alpha$. Since $\tilde{\mathcal{M}} \models \Sigma$, we have two possibilities:

a. $\tilde{\mathcal{M}}$ is an extended $\mathcal{SRIQK}_{\mathcal{NF}} \backslash U$ model of Σ. This leads to a contradiction to the assumption that $\Sigma \mid\models_e \alpha$

b. There is an extended $\mathcal{SRIQK}_{\mathcal{NF}} \backslash U$ model $\tilde{\mathcal{M}}'$ of Σ such that $\tilde{\mathcal{M}} \subset \tilde{\mathcal{M}}'$. But then this would mean $\tilde{\mathcal{M}}' \not\models \alpha$ as $\tilde{\mathcal{M}} \not\models \alpha$. This again leads to a contradiction.

Hence $\Sigma \mid\models \alpha$ whenever $\Sigma \mid\models_e \alpha$.

For the only if part, suppose that $\Sigma \not\mid\models_e \alpha$. It means that there is an extended $\mathcal{SRIQK}_{\mathcal{NF}} \backslash U$ model $\tilde{\mathcal{M}}$ of Σ such that $\tilde{\mathcal{M}} \not\models \alpha$. Let $\tilde{\mathcal{M}}' \subset \tilde{\mathcal{M}}$ such that each extended interpretation $\tilde{\mathcal{I}}$ in $\tilde{\mathcal{M}}'$ is such that $\Delta^{\tilde{\mathcal{I}}}$ is countably infinite set and $\varphi_{\tilde{\mathcal{I}}}$ is a one-to-one mapping. It is easy to show that that $\tilde{\mathcal{M}}'$ is non-empty.

Now let $\mathcal{M}$ be the set of interpretation based on $\tilde{\mathcal{M}}'$. By Corollary 6 we thus have that $\mathcal{M} \models \Sigma$ and $\mathcal{M} \not\models \alpha$. Again we have two possibilities:

- $\mathcal{M}$ is an $\mathcal{SRIQK}_{\mathcal{NF}} \backslash U$ model of Σ i.e., it is the maximal set of interpretation such that $\mathcal{M} \models \Sigma$. This leads to a contradiction as by assumption $\Sigma \mid\models \alpha$ but $\mathcal{M} \not\models \alpha$.
- $\mathcal{M}$ is not such a maximal set. But since $\mathcal{M} \models \Sigma$ there must be some $\mathcal{SRIQK}_{\mathcal{NF}} \backslash U$ model $\mathcal{M}'$ of Σ such that $\mathcal{M} \subset \mathcal{M}'$. But again this leads to a contradiction as $\mathcal{M} \not\models \alpha$ and hence $\mathcal{M}' \not\models \alpha$ whereas by assumption we have that $\Sigma \models \alpha$. Hence it must be the case that $\Sigma \mid\models_e \alpha$ whenever $\Sigma \models \alpha$.

This proves the theorem. □

Next we discuss the differences between the traditional and extended semantics from a first-order modal logic perspective.

MKNF-DL as First-order Modal Logic

First-order modal logic (FOML) extends first-order logic with modal operators (Fitting and Mendelsohn 1998; Blackburn, Wolter, and van Benthem 2006). Several assumptions need to be made when defining semantics of FOML. Based on these assumption, the semantics exhibit different characteristics. We refer to (Fitting and Mendelsohn 1998, Blackburn, Wolter, and van Benthem 2006) for further detail.

Constant vs Varying Domain One of the basic question one may ask regarding the semantics of FOML is about the domains of the interpretations (possible worlds) in consideration. *Do the domains vary across different worlds?* . In *a varying domain* semantics, different domains are assumed for different worlds whereas in *constant domain* semantics the interpretation domains across the worlds is fixed.

(non) Rigidity of Constants *Does the interpretation of constant vary across the world?*. We say the FOML semantics employ the *rigid constant assumption* if for any two worlds w and w' in a given FOML interpretation and any constant c, we have that $c^{I_w} = c^{I_{w'}}$. If this is not the case, we say the semantics employ the *non-rigid constant assumption*.

As MKNF-DL can been seen as a fragment of FOML with two modal operators **K** and **A** where both the operators are interpreted as the necessity operator (□). Note that the difference between **K** and **A** becomes obvious once we

Semantics	Domain	Constant's Interpretation
traditional	Constant	Rigid
extended	Varying	Non-rigid

Table 4: Semantics Comparison

introduce the notion of models where we require the maximality of the set of world for **K**. Table 4 describes which (FOML semantics) assumptions are satisfied by the semantics of MKNF-DL.

Reasoning in $\mathcal{SROIQK}_{\mathcal{NF}}$

One of the standard approaches approach to reasoning in some modal nonmonotonic logic is by reducing reasoning problem into several reasoning steps in some standard non-modal logic. We take a similar approach for reasoning in $\mathcal{SROIQK}_{\mathcal{NF}}$. We follow the notions presented in (Donini, Nardi, and Rosati 2002) mainly. However, due to the different semantics, proofs presented in (Donini, Nardi, and Rosati 2002) are not applicable to our approach as is.

Note that each extended model of a $\mathcal{SROIQK}_{\mathcal{NF}}$ knowledge base Σ is just a set of extended interpretations, which in themselves are standard interpretation with abstraction mapping. To this end we present the notion of $\mathcal{SROIQ}$ representability.

Definition 8. *A set of extended interpretations $\tilde{\mathcal{M}}$ is $\mathcal{SROIQ}$ representable if and only if there is an $\mathcal{SROIQ}$ knowledge base Σ such that*

$$\tilde{\mathcal{M}} = \{\tilde{\mathcal{I}} | \tilde{\mathcal{I}} \in (\mathcal{E}(\mathsf{mod}(\Sigma))\}$$

where $\mathsf{mod}(\Sigma)$ *represents the set of all $\mathcal{SROIQ}$ models of Σ. If there is no such Σ we say $\tilde{\mathcal{M}}$ is $\mathcal{SROIQ}$ unrepresentable.*

In general there are $\mathcal{SROIQK}_{\mathcal{NF}}$ knowledge bases with models that are not $\mathcal{SROIQ}$ representable.

Theorem 9. *The models of a $\mathcal{SROIQK}_{\mathcal{NF}}$ KB are in general not $\mathcal{SROIQ}$ representable.*

Proof (*proof sketch*)

From (Donini, Nardi, and Rosati 2002) we know that $\mathcal{ALCK}_{\mathcal{NF}}$ is not first-order representable and thus not $\mathcal{ALC}$ representable. Now $\mathcal{ALCK}_{\mathcal{NF}}$ is a fragment of $\mathcal{SROIQK}_{\mathcal{NF}}$ and by Theorem 7 we now that the entailment relation $\mid\models$ in the traditional semantics and $\mid\models_e$ in the extended semantics are compatible. Hence the example in the proof presented in (Donini, Nardi, and Rosati 2002) can serve as a counter example of $\mathcal{SROIQK}_{\mathcal{NF}}$ being $\mathcal{SROIQ}$ un-representable. □

Besides the result in the above theorem, we will see that for certain $\mathcal{SROIQK}_{\mathcal{NF}}$ KBs, the models are $\mathcal{SROIQ}$ representable. The question now is how to identify such $\mathcal{SROIQK}_{\mathcal{NF}}$ KBs. For this we need to understand why models of certain $\mathcal{SROIQK}_{\mathcal{NF}}$ KBs are not $\mathcal{SROIQ}$ representable. The basic idea is that we have two types of quantifiers:

1. $\exists$ and $\forall$ which quantify over the elements of the domain of interpretations

2. **K** and **A** which quantify over the interpretations themselves

It is this interaction of these quantifiers that makes models of a given knowledge base $\mathcal{SROIQ}$ unrepresentable. For example, in the concept $\exists R.\mathbf{K}D$ with R a role name and D a concept name, $\exists R$ interacts with $\mathbf{K}D$. Lets consider a set of extended interpretation $\tilde{\mathcal{M}}$ and let $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$. Now in order for the concept to be satisfiable in $(\tilde{\mathcal{I}}, \tilde{\mathcal{M}}, \tilde{\mathcal{M}})$ there should be some $x \in \Delta^{\tilde{\mathcal{I}}}$ for which we have some $y \in \Delta^{\tilde{\mathcal{I}}}$ such that $\varphi_{\tilde{\mathcal{I}}}(c) = y$ for an abstract name $c \in N_I \cup \mathbb{N}$. By semantics it is also required that c is the abstract name for some element in each world which has the property D. Such an interaction of x of one world with elements in all the possible worlds can not be expressed in first-order logic and therefore not in $\mathcal{SROIQ}$. Rather first-order interpretation expresses properties that are local and are not across the worlds.

Note that for an extended $\mathcal{SROIQK}_{\mathcal{NF}}$ model $\tilde{\mathcal{M}}$ of KB, the $\mathcal{SROIQ}$ reprsentabilty requires the existence of a $\mathcal{SROIQK}$ KB $\mathcal{O}$ such $\tilde{\mathcal{M}} = \mathcal{E}(\mathsf{mod}(\mathcal{O}))$. Hence, it is important to understand what $\mathcal{SROIQK}_{\mathcal{NF}}$ axioms are satisfied in the set of extended interpretation $\mathcal{E}(\mathsf{mod}(\mathcal{O}))$ for arbitrary $\mathcal{SROIQ}$ KB $\mathcal{O}$. The following two lemmata provides us with an insight. Note that we consider **K** only. The reason is simply that **K** and **A** are treated equivalently on the right hand side of the relation $\models$ (Donini, Nardi, and Rosati 2002).

Lemma 10. *Let $\mathcal{O}$ be a $\mathcal{SROIQ}$ knowledge base and C a $\mathcal{SROIQK}_{\mathcal{NF}}$ concept of the form $= \mathbf{K}D$ or $= \mathbf{A}D$ with D is $\mathcal{SROIQ}$ concept i.e., is* **K** *and* **A** *free. Let $\tilde{\mathcal{M}}$ be the set with*

$$\tilde{\mathcal{M}} = \mathcal{E}(\mathsf{mod}(\mathcal{O}))$$

Then for any extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$ and $x \in \Delta^{\tilde{\mathcal{I}}}$, we have that $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{M}}}$ exactly if one of the following is the case:

1. *$\mathcal{O} \models \top \sqsubseteq D$, or*
2. *$x = a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{M}}}$ and $\mathcal{O} \models D(a)$ for an individual name $a \in N_I$.*

Intuitively, this lemma ensures that the extension of a concept that is preceded by **K** can only contain named individuals unless it comprises the whole domain. A somewhat similar but intricate case is with the roles.

Lemma 11. *Let $\mathcal{O}$ be a $\mathcal{SROIQ}$ knowledge base and let R be a $\mathcal{SROIQK}_{\mathcal{NF}}$ role of the form $\mathbf{K}P$ or $\mathbf{A}P$. Further let $\tilde{\mathcal{M}} = \mathcal{E}(\mathsf{mod}(\Sigma))$. Then for any extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$ and any $x, y \in \Delta^{\tilde{\mathcal{I}}}$, we have that $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}},\tilde{\mathcal{M}}}$ exactly if one of the following holds:*

1. *$\mathcal{O} \models U \sqsubseteq P$, or*
2. *$x = a^{\tilde{\mathcal{I}}}$, $y = b^{\tilde{\mathcal{I}}}$ and $\mathcal{O} \models P(a, b)$ for some individual names $a, b \in N_I$, or*
3. *$x = a^{\tilde{\mathcal{I}}}$ and $\mathcal{O} \models \top \sqsubseteq \exists P^{-}.\{a\}$ for some individual name $a \in N_I$, or*
4. *$y = b^{\tilde{\mathcal{I}}}$ and $\mathcal{O} \models \top \sqsubseteq \exists P.\{b\}$ for some individual name $b \in N_I$, or*
5. *$x = y$ and $\mathcal{O} \models \top \sqsubseteq \exists P.\mathsf{Self}$.*

Using these observations we now define the notion of subjectively quantified $\mathcal{SROIQK}_{\mathcal{NF}}$ knowledge bases whose models are $\mathcal{SROIQ}$ representable.

Definition 12. *Let Σ be a $\mathcal{SROIQK}_{\mathcal{NF}}$ knowledge base and C be a concept expression occurring Σ in some axioms. We say Σ is subjectively quantified if and only if for each quantified sub-expression C' of C such that C' is of the form $\Xi R.D$ where $\Xi \in \{\exists, \forall, \leq n, \geq n\}$ for an non-negative integer n, then we have either*

- *R and D are $\mathcal{SROIQ}$ role and concept respectively, i.e., both are* **K** *and* **A** *free, or*
- *R is an $\mathcal{SROIQK}_{\mathcal{NF}}$ role of the form $\mathbf{K}P$ or $\mathbf{A}P$, and D is of the form $\mathbf{K}D'$, $\mathbf{A}D'$, $\neg\mathbf{K}D'$ or $\mathbf{A}D'$.* ◊

Informally, a subjectively quantified $\mathcal{SROIQK}_{\mathcal{NF}}$ knowledge base contains no concept expression in which quantified **K** and **A** free sub-expression interacts with a modalized sub-expression and vice versa. As discussed in (Donini, Nardi, and Rosati 2002) there are subjectively quantified KBs whose models cannot be represented by a finite $\mathcal{SROIQ}$ KB. From practical point of few we are interested in $\mathcal{SROIQK}_{\mathcal{NF}}$ KBs whose models can be represented by finite $\mathcal{SROIQ}$. To ensure this, we need further restrictions. A given $\mathcal{SROIQK}_{\mathcal{NF}}$ KB $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ is said to be *simple* is simple, if

1. $\mathcal{T}$ is a disjoint union of $\mathcal{T}'$ and Γ where $\mathcal{T}'$ is **K** and **A** free and $\Gamma = \mathcal{T} \setminus \Gamma$ such that axioms in Γ are of the form $\mathbf{K}C \sqsubset D$ where C is a $\mathcal{SROIQ}$ concept and D is a subjectively quantified concept.
2. for each $\mathbf{K}C \sqsubseteq D \in \Gamma$, we have that $\mathcal{T} \not\models \top \sqsubset C$
3. $\mathcal{R}$ is **K** and **A** free
4. for any $\mathcal{SROIQ}$ role P and individual name a we have that:

   a $\top \sqsubseteq \exists P^{-}.\{a\} \notin \mathcal{T}'$
   b $\top \sqsubseteq \exists P.\{a\} \notin \mathcal{T}'$
   c $\top \sqsubseteq \exists P.\mathsf{Self} \notin \mathcal{T}'$
5. $(\mathcal{T}, \mathcal{R}) \not\models U \sqsubseteq P$

Note that condition 1 and 2 similar to the conditions required for simple KBs defined in (Ke and Sattler 2008). Further condition 2, 4 and 5 ensures that whenever a model of Σ is represented by a $\mathcal{SROIQ}$ KB , then only second case in both Lemma 10 as well as in Lemma 11 holds. In the sequel we assume each $\mathcal{SROIQK}_{\mathcal{NF}}$ to be simple and subjectively quantified unless stated otherwise. Further w.log we also assume N_I to be the set of all individual names which occur in Σ.[6]

In the following we see how the models of a simple subjectively quantified $\mathcal{SROIQK}_{\mathcal{NF}}$ Σ can be represented by finite $\mathcal{SROIQ}$ knowledge bases. The standard approach is to define a set of so-called modal atoms for a given $\mathcal{SROIQK}_{\mathcal{NF}}$ KB Σ. This set can be partitioned in to the

[6] N_I can be extended whenever required.

set of atoms assumed to be true and the set of atoms assumed to false. For each such partition one can identify a $\mathcal{SROIQ}$ KB $\mathcal{O}$ such that $\mathcal{E}(\mathsf{mod}(\mathcal{O}))$ is an epistemic model of Σ.

Definition 13. *Let $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ be a simple and subjectively quantified $\mathcal{SROIQK_{NF}}$ knowledge base. Then the set of modal atoms $\mathsf{MA}(Sigma)$ of Σ as the smallest set such that:*

1. $\mathbf{K}\alpha \in \mathsf{MA}(\Sigma)$ *for each* $\alpha \in \mathcal{A}$ *with* α *a* $\mathbf{K}$ *and* $\mathbf{A}$ *free axiom,*
2. $\mathbf{K}R(a, b) \in \mathsf{MA}(\Sigma)$ *(resp.* $\mathbf{A}R(a, b) \in \mathsf{MA}(\Sigma)$*) for each* $\mathbf{K}R(a, b) \in \mathcal{A}$ *(resp.* $\mathbf{A}R(a, b) \in \mathcal{A}$*)*
3. $\mathbf{K}D(a) \in \mathsf{MA}(\Sigma)$ *for each* $a \in N_I$ *such* $\mathbf{K}C(a) \in \mathsf{MA}(\Sigma)$ *and* $\mathbf{K}C \sqsubseteq D \in \Gamma$,
4. $\mathbf{K}D(a) \in \mathsf{MA}(\Sigma)$ *(resp.* $\mathbf{A}D(a) \in \mathsf{MA}(\Sigma)$*) for each expression* $\mathbf{K}D$ *(resp.* $\mathbf{A}D$*) occurring strickly in some* C *with* $\mathbf{K}C(a) \in \mathsf{MA}(\Sigma)$ *or* $\mathbf{A}C(a) \in \mathsf{MA}(\Sigma)$ *for* $a \in N_I$,
5. $\Xi\aleph_1 R.\aleph_2 D(a) \in \mathsf{MA}(\Sigma)$ *(resp.* $\Xi\aleph_1 R.\neg\aleph_2 D(a) \in \mathsf{MA}(\Sigma)$*) and* $\aleph_1 R(a, b), \aleph_2 D(b) \in \mathsf{MA}(\Sigma)$ *for each* $b \in N_I$ *whenever the concept expression* $\Xi\aleph_1 R.\aleph_2 D$ *(resp.* $\Xi\aleph R.\neg\aleph_2 D$*) occurs strictly in some expression* C *such that* $\mathbf{K}C(a) \in \mathsf{MA}(\Sigma)$ *or* $\mathbf{A}C(a) \in \mathsf{MA}(\Sigma)$

where $\Xi \in \{\exists, \forall, \leq n, \geq n\}$ *for some non-negative integer* n *and* $\aleph_1, \aleph_2 \in \{\mathbf{K}, \mathbf{A}\}$.

Note that unlike (Donini, Nardi, and Rosati 2002) the set $\mathsf{MA}(\Sigma)$ of modal atoms in our case is finite. The reason is simply that the set N_I is finite as well as Σ itself is finite. This ensures the $\mathcal{SROIQ}$ representability of the models of Σ by finite $\mathcal{SROIQ}$ KBs.

Definition 14. *For a given $\mathcal{SROIQK_{NF}}$ KB $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$, let $[P, N]$ be a partition of $\mathsf{MA}(\Sigma)$. Further, let $C(a)$ be a $\mathcal{SROIQK_{NF}}$ axiom in $\mathcal{A}$. By $C(a)[P, N]$ we mean the expression obtained as following:*

- *replace each strict occurrence of a concept expression* $\mathbf{K}D$ *(resp.* $\mathbf{A}D$*) in* C *with* $\top$ *if* $\mathbf{K}D(a) \in P$ *(resp.* $\mathbf{A}D(a) \in P$*) else replace it with* $\bot$
- *replace each strict occurrence of a concept expression* $\Xi\aleph_1 R.\aleph_2 D$ *in* C *with* $\top$ *if* $\Xi\aleph_1 R.\aleph_2 D(a) \in P$ *else replace it with* $\bot$

where $\Xi \in \{\exists, \forall, \leq n, \geq n\}$ *for a non-negative integer* n *and* $\aleph_1 \in \{\mathbf{K}, \mathbf{A}\}$ *and* $\aleph_2 \in \{\mathbf{K}, \neg\mathbf{K}, \mathbf{A}, \neg\mathbf{A}\}$. ◊

Note that $C(a)[P, N]$ is a $\mathbf{K}$ and $\mathbf{A}$ free expression. In a similar manner one can replace modal atoms in a given knowledge base. Consequently we get a $\mathcal{SROIQ}$ knowledge base i.e., a KB without $\mathbf{K}$ and $\mathbf{A}$ operator. In the following we will see that such KBs can be used to represent $\mathcal{SROIQK_{NF}}$ models of a $\mathcal{SROIQK_{NF}}$ KB. Similar to (Donini, Nardi, and Rosati 2002),

Definition 15. *For a given $\mathcal{SROIQK_{NF}}$ Σ and a partition $[P, N]$ of $\mathsf{MA}(\Sigma)$, the $\mathcal{SROIQ}$ KB $\mathsf{Ob}_\mathbf{K}$ and $\mathsf{Ob}_\mathbf{A}$ are defined as $\mathsf{Ob}_\mathbf{K} = (\mathcal{T}', \mathcal{R}, \mathcal{A}')$ and $\mathsf{Ob}_\mathbf{A} = (\mathcal{T}', \mathcal{R}, \mathcal{A}'')$ where*

$$\mathcal{A}' = \{C(a)[P, N] | \mathbf{K}C(a) \in P\} \cup \{R(a, b) | \mathbf{K}R(a, b) \in P\}$$

and

$$\mathcal{A}'' = \{C(a)[P, N] | \mathbf{A}C(a) \in P\} \cup \{R(a, b) | \mathbf{A}R(a, b) \in P\}$$

We call $\mathsf{Ob}_\mathbf{K}$ and $\mathsf{Ob}_\mathbf{A}$ as $\mathcal{SROIQ}$ KBs corresponding to $[P, N]$. In general, not every partition is of our interest as there are partitions for which $\mathsf{Ob}_\mathbf{K}$ and $\mathsf{Ob}_\mathbf{A}$ may not lead to a model. We rather are interested in partitions that are consistent in the following sense.

Definition 16. *Let Σ be a $\mathcal{SROIQK_{NF}}$ KB. A partition $[P, N]$ of $\mathsf{MA}(\Sigma)$ is said to consistent if it satisfies:*

- *(i) if* $C(a) \in \mathcal{A}$ *(resp.* $R(a, b) \in \mathcal{A}$*), then* $\mathbf{K}C(a) \in P$ *(resp.* $\mathbf{K}R(a, b) \in P$*).*
- *(ii)* $\mathsf{Ob}_\mathbf{K}$ *is satisfiable,*
- *(iii)* $\mathsf{Ob}_\mathbf{A}$ *is satisfiable,*
- *(iv)* $\mathsf{Ob}_\mathbf{K} \not\models C(a)[P, N]$ *for each* $\mathbf{K}C(a) \in N$,
- *(v)* $\mathsf{Ob}_\mathbf{K} \not\models R(a, b)$ *for each* $\mathbf{K}R(a, b) \in N$,
- *(vi)* $\mathsf{Ob}_\mathbf{A} \not\models C(a)[P, N]$ *for each* $\mathbf{A}C(a) \in N$,
- *(vii)* $\mathsf{Ob}_\mathbf{A} \not\models R(a, b)$ *for each* $\mathbf{A}R(a, b) \in N$,
- *(viii) for each* $\Xi\aleph_1 R.\aleph_2 D(a) \in P$ *(resp.* $\Xi\aleph_1 R.\neg\aleph_2 D(a) \in P$*), we have*
  - *if* $\Xi = \exists$ *then there is some* $b \in N_I$ *such that* $\aleph_1 R(a, b) \in P$ *and* $\aleph_2 D(y) \in P$ *(resp.* $\aleph_2 \in N$*)*
  - *if* $\Xi = \forall$ *then for every* $b \in N_I$, $\aleph_1 R(a.b) \in P$ *implies that* $\aleph_2 D(b) \in P$ *(resp.* $\aleph_2 D(b) \in N$*),*
  - *if* $\Xi = \leq n$ *for a non-negative integer* n*, then there are pair-wise distinct* $b_1, \ldots, b_k \in N_I$ *with* $k \leq n$ *such that* $\aleph_1 R(a, b_i) \in P$ *and* $\aleph_2 D(b_i) \in P$ *(resp.* $\aleph_2 D(b_i) \in N$*) for* $1 \leq i \leq k$,
  - *if* $\Xi = \geq n$ *for a non-negative integer* n*, then there are pair-wise distinct* $b_1, \ldots, b_k \in N_I$ *with* $k \geq n$ *such that* $\aleph_1 R(a, b_i) \in P$ *and* $\aleph_2 D(b_i) \in P$ *(resp.* $\aleph_2 D(b_i) \in N$*) for* $1 \leq i \leq k$,
  - *for each* $\mathbf{K}C \sqsubseteq D \in \Gamma$, $\mathbf{K}D(a) \in P$ *whenever* $\mathbf{K}C(a) \in P$ *for each* $a \in N_I$. ◊

The notion of consistent partition is sufficient in the sense that only consistent partitions satisfying some additional conditions (to be discussed later) lead to $\mathcal{SROIQ}$ KBs for representing models of a given $\mathcal{SROIQK_{NF}}$ KB Σ. However, we want the other direction to be true as well i.e., for every epistemic model $\tilde{\mathcal{M}}$ of Σ there must exist a consistent partition such that $\tilde{\mathcal{M}}$ is representable by corresponding KBs.

Definition 17. *Let $(\tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$ be a pair of set of extended interpretations, then $(\tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$ induces a partition (P, N) of $\mathsf{MA}(\Sigma)$ such that $P = \{\alpha \in \mathsf{MA}(\Sigma) | (\tilde{\mathcal{M}}, \tilde{\mathcal{M}}') \models \alpha\}$. Obviously $N = \mathsf{MA}(\Sigma) \backslash P$.* ◊

We now observe the following relationship between a pair of set of extended interpretations and the partition induced by the pair.

Lemma 18. *For a given $\mathcal{SROIQK_{NF}}$ KB Σ, let $(\tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$ be a pair of set of extended interpretations, such that $(\tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$ satisfies Σ. Further let $[P, N]$ be the partition induced by $(\tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$. Then,*

- *for each* $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$ *we have* $\tilde{\mathcal{I}} \models \mathsf{Ob}_\mathbf{K}$ *and for each* $\tilde{\mathcal{I}}' \in \tilde{\mathcal{M}}'$ *we have that* $\tilde{\mathcal{I}}' \models \mathsf{Ob}_\mathbf{A}$.
- $[P, N]$ *is a consistent partition.*

Algorithm 1 isConsistent(Σ)

Require: a $\mathcal{SROIQK}_{\mathcal{NF}}$ KB Σ,
Ensure: returns true if Σ is consistent and false otherwise.

guess a partition $[P, N]$ of $\mathsf{MA}(\Sigma)$ such that:
1. $[P, N]$ is consistent
2. $\mathsf{Ob}_{\mathbf{K}}[P, N] \not\models C(a)[P, N]$ for each $\mathbf{A}C(a) \in N$
3. $\mathsf{Ob}_{\mathbf{K}}[P, N] \not\models R(a, b)$ for each $\mathbf{A}R(a, b) \in N$
4. $\mathsf{Ob}_{\mathbf{K}} = \mathsf{Ob}_{\mathbf{A}}$
5. If for each partition $[P', N']$ of $\mathsf{MA}((\mathcal{T}, \mathcal{R}, \mathcal{A}'))$ where

$$\mathcal{A}' = \mathcal{A}' \cup \{\mathbf{A}C(a) | C(x) \in \mathsf{Ob}_{\mathbf{K}}[P, N]\} \cup \{\mathbf{A}R(a, b) | R(a, b) \in \mathsf{Ob}_{\mathbf{K}}[P, N]\}$$

at least one of the following does not hold
   a. $[P', N']$ is inconsistent,
   b. $\mathsf{Ob}_{\mathbf{K}}[P, N] \models \mathsf{Ob}_{\mathbf{K}}[P', N']$,
   c. $\mathsf{Ob}_{\mathbf{K}}[P', N'] \not\models \mathsf{Ob}_{\mathbf{K}}[P, N]$
   d. $\mathsf{Ob}_{\mathbf{K}}[P, N] \models \mathsf{Ob}_{\mathbf{A}}[P', N']$
6. Then **return** true

return false

Using Lemma 18 we device Algorithm 1 for checking the consistency of a given $\mathcal{SROIQK}_{\mathcal{NF}}$ KB. The following theorem ensures the correctness of the algorithm.

Theorem 19. *For a given $\mathcal{SROIQK}_{\mathcal{NF}}$ KB Σ* isConsistent(Σ) *returns* true *iff Σ is consistent.*

Proof (*proof sketch*) Note that Lemma 18 established a key result in the sense that a set $\tilde{\mathcal{M}}$ of extended interpretation satisfying a KB Σ induces a consistent partition $[P, N]$ of $\mathsf{MA}(\Sigma)$ only. Further each $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$ is such that $\tilde{\mathcal{I}} \models \mathsf{Ob}_{\mathbf{K}}[P, N]$. This allows us and each extended interpretation satisfies. In fact one can show that $\tilde{\mathcal{M}} = \mathcal{E}(\mathsf{mod}(\mathsf{Ob}_{\mathbf{K}}[P, N]))$. Hence existence of a consistent partition as in Algorithm 1 guarantees the existence of a set of extended interpretations satisfying Σ. This set thus satisfy the first condition required by the definition of $\mathcal{SROIQK}_{\mathcal{NF}}$ models of Σ. Condition 5 then checks if this set is maximal as required by the second condition in the definition of $\mathcal{SROIQK}_{\mathcal{NF}}$ models. □

As the entailment problem can be reduced to the consistency problem, Algorithm 1 can be used for deciding entailment problem as well. The computational complexity of the reasoning in $\mathcal{SROIQK}_{\mathcal{NF}}$ thus can be determined by analyzing each step in the algorithm.

Corollary 20. *Deciding the consistency and entailment problem in $\mathcal{SROIQK}_{\mathcal{NF}}$* NEXPTIME-*complete.*

Proof The set $\mathsf{MA}(\Sigma)$ is expenential in the size of Σ, hence guessing a partition of $\mathsf{MA}(\Sigma)$ can be performed in NEXPTIME. The same is true for the partition in step 5. Consistency of a partition can be checked in polynomial time. Step 4 is linear whereas the rest of the steps are standard reasoning in $\mathcal{SROIQ}$ and thus can be decided in NEXPTIME (Horrocks, Kutz, and Sattler 2006). Hence, the overall complexity of the algorithm is NEXPTIME. Now given the fact that every $\mathcal{SROIQ}$ KB is a $\mathcal{SROIQK}_{\mathcal{NF}}$ KB, we immediately get the hardness results. □

Note that (Donini, Nardi, and Rosati 2002) and (**?**) presents tableau algorithm for deciding the reasoning tasks in $\mathcal{ALCK}_{\mathcal{NF}}$. In our case, we believe a direct implementation of Algorithm 1 allows us using some highly optimized off-the-shelf reasoner as a black-box for deciding the reasoning problems. Nevertheless, several heuristics need to be invented in order to achieve practical feasibility. We leave this as a matter of the future work. Just to mention an important remark, note that guessing the partitions in Algorithm 1 is the most expensive step as it introduced non-determinism. However, one can easily reduce the number of possible partitions by enforcing the conditions required for consistent partitions. For example, for every $\mathbf{K}$ and $\mathbf{A}$ free axiom α in $\mathcal{A}$, we must have that $\mathbf{K}\alpha \in P$ for a partition P. The reason is that extended interpretation in a model of Σ has to satisfy α. Similarly, for every $a \in N_I$ with $\mathbf{K}C(a) \in P$, we add $\mathbf{K}D(a)$ to P whenever $\mathbf{K}C \sqsubseteq D \in \Gamma$. In deed, every condition in the definition of consistent partition may serve as filtration criteria for select modal atoms in P of a partition. Consequently, we reduce the number of candidate partition and hence the overall computation time.

Conclusion and Outlook

We have seen that some features of DLs like $\mathcal{SROIQ}$ are expressive enough to cause problem when traditional semantics is applied for their MKNF extension. We thus have suggested the recently introduced semantics for $\mathcal{SROIQK}$ for expressive MKNF-DL. Later we showed the compatibility of both the semantics for DLs where the traditional semantics is applicable. This showed that for MKNF-DL upto $\mathcal{SRIQK}_{\mathcal{NF}} \setminus U$ the entailment under both the traditional semantics and the extended semantics coincides. Meanwhile we provided a comparison between the semantics from the first-order logic perspective as well, In the traditional semantics we have the assumption of constant domain along with the rigidity of constants whereas in the extended semantics we have the assumption of varying domain along with the non-rigidity of constants. Finally we have devised an algorithm for deciding reasoning problems in $\mathcal{SROIQK}_{\mathcal{NF}}$. Consequently, we showed that the time complexity of reasoning in $\mathcal{SROIQK}_{\mathcal{NF}}$ coincides with that of standard $\mathcal{SROIQ}$.

As avenues for future research, we will first implement Algorithm 1 into a practical system. For this we need to invent several heuristics in order to improvise the run time of the system. We have already pointed out some ideas in the previous section. The only way of measuring the practicability of our system is through evaluation. For this we will look for real-life applications to advocate the need for extended features we obtained via $\mathbf{K}$ and $\mathbf{A}$ operators.

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