Epistemic Querying of OWL Knowledge Bases

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Abstract. Epistemic querying extends standard ontology inferencing by allowing for deductive introspection. We propose a technique for epistemic querying of OWL 2 ontologies not featuring nominals and universal roles by a reduction to a series of standard OWL 2 reasoning steps thereby enabling the deployment of off-the-shelf OWL 2 reasoning tools for this task. We prove formal correctness of our method, justify the omission of nominals and universal role, and provide an implementation as well as evaluation results.

1 Introduction

Ontologies play a crucial role in the Semantic Web and the Web Ontology Language (OWL, [9]) is the currently single most important formalism for web-based semantic applications. OWL 2 DL – the most comprehensive version of OWL that still allows for automated reasoning – is based on the description logic (DL) $\mathcal{SROIQ}$ [6]. Querying ontologies by means of checking entailment of axioms or instance retrieval is a crucial and prominent reasoning task in semantic applications. Despite being an expressive formalism, these standard querying capabilities with OWL ontologies lack the ability for introspection (i.e., asking what the knowledge base "knows" *within* the query language). Autoepistemic DLs cope with this problem and have been investigated in the context of OWL and Semantic Web. In particular, they allow for introspection of the knowledge base in the query language by means of epistemic operators, such as the **K**-operator (paraphrased as "known to be") that can be applied to concepts and roles.

The **K**-operator allows for epistemic querying. E.g., in order to formulate queries like "known white wine that is not known to be produced in a French region" we could do an instance retrieval w.r.t. the DL concept

$$\mathbf{K}\,\mathit{WhiteWine} \sqcap \neg\exists\mathbf{K}\mathit{locatedIn}.\{\mathit{FrenchRegion}\}.$$

This can e.g. be used to query for wines that aren't explicitly excluded from being French wines but for which there is also no evidence of being French wines either (neither directly nor indirectly via deduction). For the knowledge base containing

$$\{\,\mathit{WhiteWine}(\mathit{MountadamRiesling}),\,\mathit{locatedIn}(\mathit{MountadamRiesling},\,\mathit{AustralianRegion})\}$$

the query would yield *MountadamRiesling* as a result, since it is known to be a white wine not known to be produced in a France, while a similar query without epistemic operators would yield an empty result. Hence, in the spirit of nonmonotonicity, more instances can be retrieved (and thus conclusions can been drawn) than with conventional queries in this way. Another typical use case is integrity constraint checking: testing whether the axiom

$$\mathbf{K}\,\mathit{Wine} \sqsubseteq \exists\mathbf{K}\mathit{hasSugar}.\{\mathit{Dry}\} \sqcup \exists\mathbf{K}\mathit{hasSugar}.\{\mathit{OffDry}\} \sqcup \exists\mathbf{K}\mathit{hasSugar}.\{\mathit{Sweet}\}$$

is entailed allows to check whether for every named individual that is known to be a wine it is also known (i.e. it can be logically derived from the ontology) what degree of sugar it has.[1]

However, epistemic operators (or other means for nonmonotonicity) have not found their way into the OWL specification and current reasoners do not support this feature; former research has been focused on extending tableaux algorithms for less expressive formalisms than OWL and have not paced up with the development of OWL reasoners towards optimized tableaux for expressive languages; in particular, some expressive features like nominals require special care when combined with the idea of introspection by epistemic operators.

In this paper, we take a different approach to make epistemic querying possible with OWL ontologies; namely, we reuse existing OWL reasoners in a black box fashion while providing a mechanism for reducing the problem of epistemic querying to standard DL instance retrieval; our approach reduces occurrences of the **K**-operator to introspective look-ups of instances of a concept by calls to a standard DL reasoner, while we keep the number of such calls minimal; we have implemented this approach in form of a reasoner that accepts epistemic queries and operates on non-epistemic OWL ontologies

Our contributions are the following:

[1] Note that this cannot be taken for granted even if $\mathit{Wine} \sqsubseteq \exists\mathit{hasSugar}.\{\mathit{Dry}\} \sqcup \exists\mathit{hasSugar}.\{\mathit{OffDry}\} \sqcup \exists\mathit{hasSugar}.\{\mathit{Sweet}\}$ is stated in (or can be derived from) the ontology.

- We introduce a transformation of epistemic queries to semantically identical non-epistemic queries by making introspective calls to a standard DL reasoner and by propagating the respective answer sets as nominals to the resulting query.
- We prove the correctness of this transformation in the light of some difficulties that occur with the common domain and rigid term assumptions that underly autoepistemic DLs.
- We present an efficient algorithm for implementing the above transformation with a minimal number of calls to a standard DL reasoner for the introspective look-ups of instances.
- Based on this algorithm, we provide a reasoner capable of answering epistemic queries by means of reduction to standard DL reasoning in the framework of the OWL-API extended by constructs for epistemic concepts and roles to be used in epistemic queries. First experiments show that our approach to epistemic querying is practically feasible.

The rest of this paper is structured as follows: Section 2 puts our approach into context with related work. Section 3 introduces the description logic $\mathcal{SROIQ}$ and its extension with the epistemic operator K. In Section 4, we provide the formal justification for our method of reducing $\mathcal{SROIQK}$ axiom entailment from $\mathcal{SRIQ}$ knowledge bases. In Section 5, we describe principle problems arising from allowing the use nominals or universal role in the knowledge base. In Section 6, we discuss the implementation issues and some evaluation results. We conclude in Section 7.

2 Related Work

In the early 80s Hector J. Levesque argued for the need for a richer query language in knowledge formalisms [7]. He describes that the approach to knowledge representation should be functional rather than structural and defends the idea of extending a querying language by the attribute *knows* denoted by **K** (a modality in Modal Logic terminology). In [10], Raymond Reiter makes a similar argument of in-adequacy of the standard first-order language for querying. Nevertheless, he discusses this issue in the context of databases. Similar lines of argumentation can be seen in the DL-community as well [4, 5, 3, 2] where several extensions of DLs have been presented as well as algorithms for deciding the reasoning services in such extensions. In [8], a hybrid formalism is presented which integrates Dls and rules. It is shown how epistemic querying to DL knowledge bases can be formalized in such a formalism. *EQL-Lite($\mathcal{Q}$)* is presented in [1] as a very general and powerful query language for DLs. By considering $\mathcal{SROIQK}$ as the parameter language of *EQL-Lite*, one can use their technique (by reducing into FOL queries) for answering epistemic queries. Indeed, there is a strong relationship between Theorem 6 in [1] and Lemma 16 of this work. The extension of the DL $\mathcal{ALC}$ [11] by the epistemic operator K called $\mathcal{ALCK}$, is presented in [4]. A tableau algorithm has been designed for deciding the satisfiability problem. Answering queries in $\mathcal{ALCK}$ put to $\mathcal{ALC}$ knowledge bases is also discussed.

In this work we mainly focus on DLs extended with the epistemic operator K following notions presented in [4]. However, we consider more expressive DLs rather than just $\mathcal{ALC}$.

3 Preliminaries

In this section, we present an introduction to the description logic $\mathcal{SROIQ}$ and its extension with the epistemic operator K.

3.1 Description Logics $\mathcal{SROIQ}$

We start by presenting the syntax and semantics of $\mathcal{SROIQ}$. It is an extension of $\mathcal{ALC}$ with inverse roles($\mathcal{I}$), role hierarchies($\mathcal{H}$), nominals($\mathcal{O}$) and qualifying number restrictions($\mathcal{Q}$). Besides it also allows for several role constructs and axioms.

Definition 1. For the signature of $\mathcal{SROIQ}$ we have finite and disjoint sets N_C, N_R and N_I of *concept names*, *role names* and *individual names* respectively.[2] Further the set N_R is partitioned into two sets namely, $\mathbf{R_s}$ and $\mathbf{R_n}$ of *simple* and *non-simple* roles respectively. The set $\mathbf{R}$ *of* $\mathcal{SROIQ}$*-roles* is

$$\mathbf{R} := U \mid N_R \mid N_R^-$$

where U is called the *universal role*. Further, we define a function Inv on roles such that $\mathsf{Inv}(R) = R^-$ if R is a role name, $\mathsf{Inv}(R) = S$ if $R = S^-$ and $\mathsf{Inv}(U) := U$.

The set of $\mathcal{SROIQ}$-concepts is the smallest set satisfying the following properties:

- every concept name $A \in N_C$ is a concept;
- $\top$(top concept) and $\bot$ (bottom concept) are concept;

[2] Finiteness, in particular for N_I, is required for the further considerations. However note that the signature is not bounded and can be extended whenever this should be necessary.

– if C, D are concepts, R is a role, S is a simple role, $a_1, \ldots, a_n$ are individual names and n a non-negative integer then following are concepts:

$$
\begin{array}{rl}
\neg C & \text{(negation)} \\
\exists S.\mathsf{Self} & \text{(self)} \\
C \sqcap D & \text{(conjunction)} \\
C \sqcup D & \text{(disjunction)} \\
\forall R.C & \text{(universal quantification)} \\
\exists R.C & \text{(existential quantification)} \\
\leq nS.C & \text{(at least number restriction)} \\
\geq nS.C & \text{(at most number restriction)} \\
\{a_1, \ldots, a_n\} & \text{(nominals / one-of)}
\end{array}
$$

An *RBox axiom* is an expression of one the following forms:

1. $R_1 \circ \cdots \circ R_n \sqsubseteq R$ where $R_1, \ldots, R_n, R \in \mathbf{R}$ and if $n = 1$ and $R_1 \in \mathbf{R_s}$ then $R \notin \mathbf{R_n}$,
2. $\mathsf{Ref}(R)$ (reflexivity), $\mathsf{Tra}(R)$ (transitivity), $\mathsf{Irr}(R)$ (irreflexivity), $\mathsf{Dis}(R, R')$ (role disjointness), $\mathsf{Sym}(R)$ (Symmetry), $\mathsf{Asy}(R)$ (Asymmetry) with $R, R \in \mathbf{R}$.

RBox axioms of the first form i.e., $R_1 \circ \cdots \circ R_n \sqsubseteq R$ are called *role inclusion axioms* (RIAs). An RIA is *complex* if $n > 1$. Whereas the RBox axioms of the second form e.g., $\mathsf{Ref}(R)$, are called *role characteristics*. A $\mathcal{SROIQ}$-RBox $\mathcal{R}$ is a finite set of RBox axioms such that the following conditions are satisfied:[3]

– there is a strict (irreflexive) total order $\prec$ on $\mathbf{R}$ such that
  - for $R \in \{S, \mathsf{Inv}(S)\}$, we have that $S \prec R$ iff $\mathsf{Inv}(S) \prec R$ and
  - every RIA is of the form $R \circ R \sqsubseteq R$, $\mathsf{Inv(R)} \sqsubseteq R$, $R_1 \circ \ldots R_n \sqsubseteq R$, $R \circ R_1 \circ \cdots \circ R_n \sqsubseteq R$ or $R_1 \circ \cdots \circ R_n \circ R \sqsubseteq R$ where $R, R_1 \ldots, R_n \in \mathbf{R}$ and $R_i \prec R$ for $1 \leq i \leq n$.

– any role characteristic of the form $\mathsf{Irr}(S)$, $\mathsf{Dis}(S, S')$ or $\mathsf{Asy}(S)$ is such that $S, S' \in \mathbf{R_s}$ i.e., we allow only for simple role in these role characteristics.

A $\mathcal{SROIQ}$ *general concept inclusion axiom* (GCI) is an expression of the form $C \sqsubseteq D$, where C and D are $\mathcal{SROIQ}$-concepts. A $\mathcal{SROIQ}$*-TBox* is a finite set of $\mathcal{SROIQ}$-GCIs.

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[3] These conditions are enforced to attain decidability. We usually call an RBox to be *regular* because of the first condition.

An $\mathcal{SROIQ}$-*ABox axiom* is of the form $C(a)$, $R(a,b)$, $a \doteq b$ or $a \neq b$ for the individual names a and b, $\mathcal{SROIQ}$-role R and a $\mathcal{SROIQ}$-concept C. A $\mathcal{SROIQ}$-*ABox* is a finite set of $\mathcal{SROIQ}$-ABox axioms.

A $\mathcal{SROIQ}$-*knowledge base* is a tuple $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ where $\mathcal{T}$ is a $\mathcal{SROIQ}$-TBox, $\mathcal{SROIQ}$-$\mathcal{R}$ is a role hierarchy and $\mathcal{SROIQ}$-$\mathcal{A}$ is a ABox. $\Diamond$

To define the semantics of $\mathcal{SROIQ}$, we introduce the notion of interpretations.

Definition 2. A $\mathcal{SROIQ}$-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is composed of a non-empty set $\Delta^{\mathcal{I}}$ called the *domain of* $\mathcal{I}$ and a *mapping function* $\cdot^{\mathcal{I}}$ such that:

- $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ for every concept name A;
- $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for every $R \in N_R$;
- $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for every individual name a.

Further the universal role U is interpreted as a total relation on $\Delta^{\mathcal{I}}$ i.e., $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The bottom concept $\bot$ and top concept $\top$ are interpreted by $\emptyset$ and $\Delta^{\mathcal{I}}$ respectively. Now the mapping $.^{\mathcal{I}}$ is extended to roles and concepts as follows:

$$\begin{aligned}
(R^-)^{\mathcal{I}} &= \{(x,y) \mid (y,x) \in R^{\mathcal{I}}\} \\
(\neg C)^{\mathcal{I}} &= \Delta \setminus C^{\mathcal{I}} \\
(\exists S.\mathsf{Self})^{\mathcal{I}} &= \{x \mid (x,x) \in S^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{p_1 \in \Delta \mid \forall p_2.(p_1,p_2) \in R^{\mathcal{I}} \rightarrow p_2 \in C^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &= \{p_1 \in \Delta \mid \exists p_2.(p_1,p_2) \in R^{\mathcal{I}} \wedge p_2 \in C^{\mathcal{I}}\} \\
(\leq nS.C)^{\mathcal{I}} &= \{p_1 \in \Delta \mid \#\{p_2 \mid (p_1,p_2) \in S^{\mathcal{I}} \wedge p_2 \in C^{\mathcal{I}}\} \leq n\} \\
(\geq nS.C)^{\mathcal{I}} &= \{p_1 \in \Delta \mid \#\{p_2 \mid (p_1,p_2) \in S^{\mathcal{I}} \wedge p_2 \in C^{\mathcal{I}}\} \geq n\} \\
\{a_1, \ldots, a_n\}^{\mathcal{I}} &= \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}
\end{aligned} \qquad \Diamond$$

where C, D are $\mathcal{SROIQ}$-concepts, R, S are roles, n is a non-negative integer and $\#M$ represents the cardinality of the set M.

Given an axiom α (TBox, RBox or ABox axiom), we say the an interpretation $\mathcal{I}$ satisfies α, written $\mathcal{I} \models \alpha$, if it satisfies the condition given in Table 1. Similarly $\mathcal{I}$ satisfies a TBox $\mathcal{T}$, written $\mathcal{I} \models \mathcal{T}$, if it satisfies all the axioms in $\mathcal{T}$. The satisfaction of an RBox and an ABox by an interpretation is defined in the same way. We say $\mathcal{I}$ satisfies a knowledge base $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ if it satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$. We write $\mathcal{I} \models \Sigma$. We call

$\mathcal{I}$ a model of Ψ. A knowledge base is said to be *consistent* if it has a model.

We now present the extension of the DL $\mathcal{SROIQ}$ by the epistemic operator K. We call this extension $\mathcal{SROIQK}$.

Table 1. Semantics of $\mathcal{SROIQ}$ axioms

Axiom α	$\mathcal{I} \models \alpha$, if
$R_1 \circ \cdots \circ R_n \sqsubseteq R$	$R_1^{\mathcal{I}} \circ \cdots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
Tra(R)	$R^{\mathcal{I}} \circ R^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
Ref(R)	$(x, x) \in R^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
Irr(S)	$(x, x) \notin S^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
Dis(S,T)	$(x, y) \in S^{\mathcal{I}}$ implies $(x, y) \notin T^{\mathcal{I}}$ for all $x, y \in \Delta^{\mathcal{I}}$
Sym(S)	$(x, y) \in S^{\mathcal{I}}$ implies $(y, x) \in S^{\mathcal{I}}$ for all $x, y \in \Delta^{\mathcal{I}}$
Asy(S)	$(x, y) \in S^{\mathcal{I}}$ implies $(x, y) \notin S^{\mathcal{I}}$ for all $x, y \in \Delta^{\mathcal{I}}$
$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
$R(a, b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
$a \doteq b$	$a^{\mathcal{I}} = a^{\mathcal{I}}$
$a \not\doteq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$

3.2 K-extensions of $\mathcal{SROIQ}$

The embedding of the epistemic operator K into the description logic $\mathcal{ALC}$ was first proposed in [3]. The logic obtained is called $\mathcal{ALCK}$. A similar approach has been taken in [4], which we follow in this work. We consider $\mathcal{SROIQ}$ as the basis DL and call its K-extension $\mathcal{SROIQK}$. In $\mathcal{SROIQK}$ we allow K in front of the concepts and role names. In the following we provide the formal syntax and semantics of such language where N_C, N_R, N_I, $\mathbf{R}$ are as in Definition 1.

Definition 3. A $\mathcal{SROIQK}$-role is defined as follows:

- every $R \in \mathbf{R}$ is a $\mathcal{SROIQK}$-role;
- if R is a $\mathcal{SROIQK}$-role than so are $\mathsf{K}R$ and R^-.

We call a $\mathcal{SROIQK}$-role an *epistemic role* if $\mathbf{K}$ occurs in it. An epistemic role is *simple* if it is of the form $\mathbf{K}S$ where S is a simple $\mathcal{SROIQ}$-role. Now $\mathcal{SROIQK}$-concepts are defined as follows:

- every $\mathcal{SROIQ}$-concept is an $\mathcal{SROIQ}$-concept;

– if C and D are $\mathcal{SROIQK}$-concepts, and S and R are $\mathcal{SROIQK}$ roles with S being simple, then the following are $\mathcal{SROIQK}$-concepts:

$$\mathbf{K}C \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall R.C \mid \exists R.C \mid \leq nS.C \mid \geq nS.C$$ ♢

The semantics of $\mathcal{SROIQK}$ is given as *possible world semantics* in terms of *epistemic interpretations*. Thereby following assumptions are made:

1. all interpretations are defined over a fixed infinite domain Δ (Common Domain Assumption);
2. for all interpretations, the mapping from individuals to domains elements is fixed: it is just the identity function (Rigid Term Assumption).

Definition 4. An epistemic interpretation for $\mathcal{SROIQK}$ is a pair $(\mathcal{I}, \mathcal{W})$ where $\mathcal{I}$ is a $\mathcal{SROIQ}$-interpretation and $\mathcal{W}$ is a set of $\mathcal{SROIQ}$-interpretations, where $\mathcal{I}$ and all of $\mathcal{W}$ have the same infinite domain Δ with $N_I \subset \Delta$. The interpretation function $\cdot^{\mathcal{I},\mathcal{W}}$ is then defined as follows:

$$\begin{aligned}
a^{\mathcal{I},\mathcal{W}} &= a \quad \text{for } a \in N_I \\
A^{\mathcal{I},\mathcal{W}} &= A^{\mathcal{I}} \quad \text{for } A \in N_C \\
R^{\mathcal{I},\mathcal{W}} &= R^{\mathcal{I}} \quad \text{for } R \in N_R \\
\top^{\mathcal{I},\mathcal{W}} &= \Delta \quad \text{(the domain of } \mathcal{I}) \\
\bot^{\mathcal{I},\mathcal{W}} &= \emptyset \\
(C \sqcap D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cap D^{\mathcal{I},\mathcal{W}} \\
(C \sqcup D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cup D^{\mathcal{I},\mathcal{W}} \\
(\neg C)^{\mathcal{I},\mathcal{W}} &= \Delta \setminus C^{\mathcal{I},\mathcal{W}} \\
(\forall R.C)^{\mathcal{I},\mathcal{W}} &= \{p_1 \in \Delta \mid \forall p_2.(p_1, p_2) \in R^{\mathcal{I},\mathcal{W}} \rightarrow p_2 \in C^{\mathcal{I},\mathcal{W}}\} \\
(\exists R.C)^{\mathcal{I},\mathcal{W}} &= \{p_1 \in \Delta \mid \exists p_2.(p_1, p_2) \in R^{\mathcal{I},\mathcal{W}} \wedge p_2 \in C^{\mathcal{I},\mathcal{W}}\} \\
(\leq nR.C)^{\mathcal{I},\mathcal{W}} &= \{d \mid \#\{e \in C^{\mathcal{I},\mathcal{W}} \mid (d, e) \in R^{\mathcal{I},\mathcal{W}}\} \leq n\} \\
(\geq nR.C)^{\mathcal{I},\mathcal{W}} &= \{d \mid \#\{e \in C^{\mathcal{I},\mathcal{W}} \mid (d, e) \in R^{\mathcal{I},\mathcal{W}}\} \geq n\} \\
(\mathbf{K}C)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J} \in \mathcal{W}} (C^{\mathcal{J},\mathcal{W}}) \\
(\mathbf{K}R)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J} \in \mathcal{W}} (R^{\mathcal{J},\mathcal{W}})
\end{aligned}$$

where C and D are $\mathcal{SROIQK}$-concepts and R is a $\mathcal{SROIQK}$-role. Further for an epistemic role $(\mathbf{K}R)^-$, we set $[(\mathbf{K}R)^-]^{\mathcal{I}} := (\mathbf{K}R^-)^{\mathcal{I}}$. ♢

From the above one can see that $\mathbf{K}C$ is interpreted as the set of objects that are in the interpretation of C under every interpretation in $\mathcal{W}$. Note that the rigid term assumption implies the unique name assumption (UNA) i.e., for any interpretation $\mathcal{I} \in \mathcal{W}$ and for any two distinct individual names a and b we have that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

The notions of GCI, assertion, role hierarchy, ABox, TBox and knowledge base, and their interpretations as defined in Definition 1 and 2 can be extended to that of $\mathcal{SROIQK}$ by allowing for $\mathcal{SROIQK}$-concepts and $\mathcal{SROIQK}$-roles in their definitions.

An *epistemic model* for a $\mathcal{SROIQK}$-knowledge base $\Psi = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ is a *maximal* non-empty set $\mathcal{W}$ of $\mathcal{SROIQ}$-interpretations such that $(\mathcal{I}, \mathcal{W})$ satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$ for each $\mathcal{I} \in \mathcal{W}$. A $\mathcal{SROIQK}$-knowledge base Ψ is said to be *satisfiable* if it has an epistemic model. The knowledge base Ψ *entails* an axiom φ, written $\Psi \Vdash \varphi$, if for every epistemic model $\mathcal{W}$ of Ψ, we have that for every $\mathcal{I} \in \mathcal{W}$, the epistemic interpretation $(\mathcal{I}, \mathcal{W})$ satisfies φ. By definition every $\mathcal{SROIQ}$-knowledge base is an $\mathcal{SROIQK}$-knowledge base. Note that a given $\mathcal{SROIQ}$-knowledge base Σ has up to isomorphism only one unique epistemic model which is the set of all models of Σ having infinite domain and satisfying the unique name assumption. We denote this model by $\mathcal{M}(\Sigma)$.

4 Deciding Entailment of Epistemic Axioms

In this section we provide a way for deciding epistemic entailment based on techniques for non-epistemic standard reasoning. More precisely, we consider the problem whether a $\mathcal{SROIQK}$ axiom α is entailed by a $\mathcal{SRIQ}$ knowledge base Σ, where $\mathcal{SRIQ}$ is defined as $\mathcal{SROIQ}$ excluding nominals and the universal role. That is, we distinguish the *querying language* from the *modeling language*. One primary use of the $\mathbf{K}$ operator that we focus on in this paper is for knowledge base introspection in the query, which justifies to exclude it from the modeling language in exchange for reducibility to standard reasoning. The reasons for disallowing the use of nominals and the universal role will be discussed in Section 5.

The basic, rather straightforward idea to decide entailment of an axiom containing $\mathbf{K}$ operators is to disassemble the axiom, query for the named individuals contained in extensions for every subexpression preceded by $\mathbf{K}$, and use the results to rewrite the axiom into one that is free of $\mathbf{K}$s. While we will show that this idea is theoretically and practically feasible, some problems need to be overcome that arise from the definition of epistemic models, in particular the rigid term assumption and the common domain assumption.

As a consequence of the rigid name assumption, every $\mathcal{I} \in \mathcal{M}(\Sigma)$ satisfies the condition that individual names are interpreted by different individuals (this condition per se is commonly referred to as the *unique* name assumption). In order to enforce this behavior (which is not ensured by the non-epistemic standard DL semantics) we have to explicitly axiomatize this condition.

Definition 5. Given a $\mathcal{SRIQ}$ knowledge base Σ, we denote by Σ_{UNA} the knowledge base $\Sigma \cup \{a \not\doteq b \mid a, b \in N_I, a \neq b\}$. $\Diamond$

Fact 6. *The set of models of Σ_{UNA} is exactly the set of those models of Σ that satisfy the unique name assumption.*

As another additional constraint on epistemic interpretations, the domain is required to be infinite (imposed by the common domain assumption). However, standard DL reasoning as performed by OWL inference engines adheres to a semantics that allows for both finite and infinite models. Therefore, in order to show that we can use standard inferencing tools as a basis of epistemic reasoning, we have to prove that finite models can be safely dismissed from the consideration, without changing the results. We obtain this result by arguing that for any finite interpretation we find an infinite one which "behaves the same" in terms of satisfaction of axioms and hence will make up for the loss of the former. The following definition and lemma provide a concrete construction for this.

Definition 7. For any $\mathcal{SRIQ}$ interpretation $\mathcal{I}$, the *lifting* of $\mathcal{I}$ to ω is the interpretation $\mathcal{I}_\omega$ defined as follows:

- $\Delta^{\mathcal{I}_\omega} := \Delta^{\mathcal{I}} \times \mathbb{N}$,
- $a^{\mathcal{I}_\omega} := \langle a^{\mathcal{I}}, 0\rangle$ for every $a \in N_I$,
- $A^{\mathcal{I}_\omega} := \{\langle x, i\rangle \mid x \in A^{\mathcal{I}} \text{ and } i \in \mathbb{N}\}$ for each concept name $A \in N_C$,
- $r^{\mathcal{I}_\omega} := \{(\langle x, i\rangle, \langle x', i\rangle) \mid (x, x') \in r^{\mathcal{I}} \text{ and } i \in \mathbb{N}\}$ for every role name $r \in N_R$. $\Diamond$

Lemma 8. *For all $\langle x, i\rangle \in \Delta^{\mathcal{I}_\omega}$ and all $\mathcal{SRIQ}$-concepts C that $\langle x, i\rangle \in C^{\mathcal{I}_\omega}$ if and only if $x \in C^{\mathcal{I}}$.*

Proof. The proof is by the induction on the structure of C:

- For the atomic concept, $\top$ or $\bot$ it follows immediately from the definition of $\mathcal{I}_\omega$.
- Let $C = \neg D$. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (\neg D)^{\mathcal{I}}$
  $\Leftrightarrow x \notin D^{\mathcal{I}}$
  $\Leftrightarrow \langle x, i\rangle \notin D^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ (Induction)
  $\Leftrightarrow \langle x, i\rangle \in (\neg D)^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$.

– Let $C = C_1 \sqcap C_2$. For any $x \in \Delta^{\mathcal{I}}$ we have that
$x \in (C_1 \sqcap C_2)^{\mathcal{I}}$
$\Leftrightarrow x \in C_1^{\mathcal{I}}$ and $x \in C_2^{\mathcal{I}}$
$\Leftrightarrow \langle x, i\rangle \in C_1^{\mathcal{I}_\omega}$ and $\langle x, i\rangle \in C_2^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ (Induction)
$\Leftrightarrow \langle x, i\rangle \in (C_1 \sqcap C_2)^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$.

– Let $C = \exists R.D$ for $R \in \mathbf{R}$. For any $x \in \Delta^{\mathcal{I}}$ we have that
$x \in (\exists R.D)^{\mathcal{I}}$
$\Leftrightarrow$ there is a $y \in \Delta^{\mathcal{I}}$ such that $(x, y) \in R^{\mathcal{I}}$ and $y \in D^{\mathcal{I}}$
$\Leftrightarrow$ there is $\langle y, i\rangle \in \Delta^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ with $(\langle x, i\rangle, \langle y, i\rangle) \in R^{\mathcal{I}_\omega}$ and $\langle y, i\rangle \in D^{\mathcal{I}_\omega}$ (Def 7 and Induction)
$\Leftrightarrow \langle x, i\rangle \in (\exists R.D)^{\mathcal{I}_\omega}$

– The rest of the cases can be proved analogously.

Lemma 9. *Let Σ be a $\mathcal{SRIQ}$ knowledge base. For any interpretation $\mathcal{I}$ we have that*

$$\mathcal{I} \models \Sigma \text{ if and only if } \mathcal{I}_\omega \models \Sigma.$$

Proof. First we note that it follows immediately from the definition of $\mathcal{I}_\omega$ that for any $\mathcal{SRIQ}$-role $R \in \mathbf{R}$ and $(\langle x, i\rangle, \langle y, i'\rangle) \in \Delta^{\mathcal{I}_\omega}$ for $i, i' \in \mathbb{N}$ we have that $(\langle x, i\rangle, \langle y, i'\rangle) \in R^{\mathcal{I}_\omega}$ if an only if $(x, y) \in R^{\mathcal{I}}$ and $i = i'$ for an interpretation $\mathcal{I}$. Now for any RIA $R_1 \circ \ldots R_n \sqsubseteq R$ we have that:
$\mathcal{I} \models R_1 \circ \ldots R_n \sqsubseteq R$
$\Leftrightarrow \mathcal{I} \models R_1^{\mathcal{I}} \circ \ldots R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$, whenever $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ then $(x_0, x_n) \in R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$ and any $j \in \mathbb{N}$, whenever $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$ for $1 \leq i \leq n$ then $(\langle x_0, j\rangle, \langle x_n, j\rangle) \in R^{\mathcal{I}_\omega}$
$\Leftrightarrow \mathcal{I}_\omega \models R_1 \circ \ldots R_n \sqsubseteq R$.
The second last equivalence holds as $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ and any non-negative integer j implies that $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$. Similary $(\langle x_{i-1}, j_{i-1}\rangle, \langle x_i, j_i\rangle) \in R^{\mathcal{I}_\omega}$ for $1 \leq i \leq n$ implies that $(x_{i-1}, x_i) \in R^{\mathcal{I}}$ and that all j_i, s are equal. And the same holds for the role R.

Similary, for any role characteristic $\mathsf{Ref}(R)$, we have that:
$\mathcal{I} \models \mathsf{Ref}(R)$
$\Leftrightarrow (x, x) \in R^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
$\Leftrightarrow (\langle x, j\rangle, \langle x, j\rangle) \in R^{\mathcal{I}_\omega}$ for any $j \in \mathbb{N}$ and $x \in \Delta^{\mathcal{I}}$
$\Leftrightarrow (\langle x, j\rangle, \langle x, j\rangle) \in R^{\mathcal{I}_\omega}$ for any $\langle x, j\rangle \in \Delta^{\mathcal{I}_\omega}$ as $\Delta^{\mathcal{I}_\omega} = \Delta^{\mathcal{I}} \times \mathbb{N}$
$\Leftrightarrow \mathcal{I}_\omega \models \mathsf{Ref}(R)$.
In the same way, we can prove for any of the rest of the role characteristics that whenever $\mathcal{I}$ models it so does $\mathcal{I}_\omega$. Consequently we have that

for any role hierarchy $\mathcal{R}$, $\mathcal{I} \models \mathcal{R}$ if and only if $\mathcal{I}_\omega \models \mathcal{R}$.

Invoking Lemma 8, we get that for any GCI $C \sqsubseteq D$ and for any interpretation $\mathcal{I}$, $C^\mathcal{I} \subseteq D^\mathcal{I}$ if and only if $C^{\mathcal{I}_\omega} \subseteq D^{\mathcal{I}_\omega}$. Further for any TBox $\mathcal{T}$, $\mathcal{I} \models \mathcal{T}$ if and only if $\mathcal{I}_\omega \models \mathcal{T}$.

Finally for an ABox $\mathcal{A}$ we show that for each assertion in $\alpha \in \mathcal{A}$, $\mathcal{I} \models \alpha$ if and only if $\mathcal{I}_\omega \models \alpha$.

- α is of the form $C(a)$: Now for an interpretation $\mathcal{I}$ it follows from the definition of $\mathcal{I}_\omega$ that $a^{\mathcal{I}_\omega} = (a^\mathcal{I}, 0)$. As we have already shown that $a^\mathcal{I} \in C^\mathcal{I}$ if and only if $(a^\mathcal{I}, i) \in C^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$. Hence we get that $a^\mathcal{I} \in C^\mathcal{I}$ if and only if $(a^\mathcal{I}, 0) \in C^{\mathcal{I}_\omega}$.
- Analogously we can show an interpretation $\mathcal{I}$ satisfies an assertion if and only if $\mathcal{I}_\omega$ does so.

The actual justification for our technique of rewriting axioms containing **K**s into **K**-free ones exploiting intermediate reasoner calls comes from the fact that (except for some remarkable special cases) the semantic extension of expressions proceeded by **K** can only contain named individuals. We prove this by exploiting certain symmetries on the model set $\mathcal{M}(\Sigma)$. Intuitively, one can freely swap or permute anonymous individuals (i.e., domain elements which do not correspond to any individual name) in a model of some knowledge base without losing modelhood, as detailed in the following definition and lemma.

Definition 10. Given an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, a set Δ with $N_I \subseteq \Delta$, and a bijection $\varphi : \Delta^\mathcal{I} \to \Delta$ with $\varphi(a^\mathcal{I}) = a$ for all $a \in N_I$, the *renaming* of $\mathcal{I}$ according to φ, denoted by $\varphi(\mathcal{I})$, is defined as the interpretation $(\Delta, \cdot^{\varphi(\mathcal{I})})$ with:

- $a^{\varphi(\mathcal{I})} = \varphi(a^\mathcal{I}) = a$ for every individual name a
- $A^{\varphi(\mathcal{I})} = \{\varphi(z) \mid z \in A^\mathcal{I}\}$ for every concept name A
- $P^{\varphi(\mathcal{I})} = \{(\varphi(z), \varphi(w)) \mid (z, w) \in P^\mathcal{I}\}$ for every role name P ◊

Lemma 11. *Let Σ be a $\mathcal{SRIQ}$ knowledge base and let $\mathcal{I}$ be a model of Σ with infinite domain. Then, every renaming $\varphi(\mathcal{I})$ of $\mathcal{I}$ satisfies $\varphi(\mathcal{I}) \in \mathcal{M}(\Sigma)$.*

Proof. By definition, the renaming satisfies the common domain and rigid term assumption. Modelhood w.r.t. Σ immediately follows from the isomorphism lemma of first-order interpretations [12] since $\mathcal{I}$ and $\varphi(\mathcal{I})$ are isomorphic and φ is an isomorphism from $\mathcal{I}$ to $\varphi(\mathcal{I})$. □

This insight can be used to "move" every anonymous individual into the position of another individual which serves as a counterexample for membership in some given concept D, unless the concept is equivalent to $\top$. This allows to prove that $\mathbf{K}D$ contains merely named individuals, given that it is not universal.

Lemma 12. *Let Σ be a $\mathcal{SHIQ}$ knowledge base. For any epistemic concept $C = \mathbf{K}D$ with $\Sigma_{\text{UNA}} \not\models D \equiv \top$ and $x \in \Delta$, we have that $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ iff x is named such that there is an individual $a \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models D(a)$.*

Proof. " $\Rightarrow$ "
Suppose that $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$. It means that

$$x \in \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

To the contrary, suppose that there is no $a \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$ and $\Sigma_{\text{UNA}} \models D(a)$ i.e., x is an anonymous element. Since $\Sigma_{\text{UNA}} \not\models \top \equiv D$, there is a model $\mathcal{I}'$ of Σ_{UNA} such that $D^{\mathcal{I}'} \neq \Delta^{\mathcal{I}'}$. This implies that there is a $y \in \Delta^{\mathcal{I}'}$ such that $y \notin D^{\mathcal{I}'}$. Considering $\mathcal{I}'_\omega$, we can invoke Lemma 9 to ensure $\mathcal{I}'_\omega \models \Sigma_{\text{UNA}}$, moreover Lemma 8 guarantees $\langle y, 1\rangle \notin D^{\mathcal{I}'_\omega}$. On the other hand, by construction, $\langle y, 1\rangle$ is anonymous. Let $\varphi : \Delta^{\mathcal{I}'} \times \mathbb{N} \to \Delta$ be a bijection such that $\varphi(a^{\mathcal{I}'_\omega}) = a^{\mathcal{I}}$ for all $a \in N_I$ and $\varphi(\langle y, 1\rangle) = x$. Such a φ exists, as $|\Delta^{\mathcal{I}'} \times \mathbb{N}| = |\Delta|$ and $\mathcal{I}'_\omega$ satisfies the unique name assumption. By Lemma 11, we get that $\varphi(\mathcal{I}'_\omega) \in \mathcal{M}(\Sigma)$. By the choice of φ we get $x \notin D^{\varphi(\mathcal{I}'_\omega)}$ due to $\langle y, 1\rangle \notin D^{\mathcal{I}'_\omega}$ and the fact that φ is an isomorphism. In particular,

$$x \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

which is a contradiction.
" $\Leftarrow$ "
Suppose there is $a \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$ and $\Sigma_{\text{UNA}} \models D(a)$. This implies that for any $\mathcal{I} \in \mathcal{M}(\Sigma)$ we have that $x \in D^{\mathcal{I}}$. Hence we get that $x \in \mathbf{K}D^{\mathcal{I},\mathcal{M}(\Sigma)}$.

A similar property can be proved for the roles as well. Before, we have to take care of the exceptional case of the universal role.

Claim 13. *Let Σ be a knowledge base. For the universal role U we have:*

$$\mathsf{K}U^{\mathcal{I},\mathcal{M}(\Sigma)} = U^{\mathcal{I},\mathcal{M}(\Sigma)}$$

The claim follows trivially as $U^{\mathcal{J}} = \Delta \times \Delta$ for any $\mathcal{J} \in \mathcal{M}(\Sigma)$. This means that $\bigcap_{\mathcal{J}\in\mathcal{M}(\Sigma)} U^{\mathcal{J}} = \Delta \times \Delta$. Thus, as in the case of concepts, whenever an epistemic concept contains a role of the form $\mathsf{K}U$, it will be simply replaced by U. That, for $\mathcal{SRIQ}$ knowledge bases, no other role than U is universal (in all models) is straightforward and can be shown using the construction from Definition 7.

We can now also show that the extension of every role preceded by K (except for the universal one), consists only of pairs of named individuals.

Lemma 14. *Let Σ be a $\mathcal{SRIQ}$ knowledge base. For any epistemic role $R = \mathsf{K}P$ with $P \neq U$, and $x, y \in \Delta$ we have that $(x, y) \in R^{\mathcal{I},\mathcal{M}(\Sigma)}$ iff at least one of the following holds:*

1. *there are individual names $a, b \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$, $b^{\mathcal{I},\mathcal{M}(\Sigma)} = y$ and $\Sigma_{\mathrm{UNA}} \models P(a, b)$.*
2. *$x = y$ and $\Sigma_{\mathrm{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$.*

Proof
" $\Leftarrow$ "
Depending on which case hold, we make the following case distinction:

– Suppose that $x = y$ and $\Sigma_{\mathrm{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$. As $\mathcal{M}(\Sigma)$ the epistemic model of Σ, therfore every interpretation in $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies the UNA and by Fact 6 we get that $\mathcal{J} \models \Sigma_{\mathrm{UNA}}$. This means for every interpretation $\mathcal{J} \in \mathcal{M}(\Sigma)$ we have that $(x', x') \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ i.e.,

$$(x', x') \in \bigcap_{\mathcal{J}\in\mathcal{M}(\Sigma)} P^{\mathcal{J},\mathcal{M}(\Sigma)}$$

for any $x' \in \Delta$. By semantics, therefore, $(x', x') \in \mathsf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ for any $x' \in \Delta$. In particular, we have that $(x, y) \in \mathsf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $x = y$.
– Suppose there are $a, b \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$, $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\mathrm{UNA}} \models P(a, b)$. By assumption we have that $\Sigma_{\mathrm{UNA}} \models P(a, b)$. Therefore, we have that $(x, y) \in P^{\mathcal{I}}$ for any interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$ as each such $\mathcal{I}$ satisfies UNA. Hence $(x, y) \in \mathsf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$.

" $\Rightarrow$ "
We first suppose that the second case of the lemma does not hold. Therefore, we have to show that there are a, b with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$, $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$

and $\Sigma_{\text{UNA}} \models P(a,b)$. To the contrary suppose that there is no such $a, b \in N_I$. We distinguish two cases.

- There are a, b with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ but $\Sigma_{\text{UNA}} \not\models P(a,b)$. Now $\Sigma \not\models P(a,b)$ implies that there is an interpretation $\mathcal{I}'$ with $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \notin P^{\mathcal{I}'}$. Considering $\mathcal{I}'_\omega$, we can invoke Lemma 9 to ensure $\mathcal{I}'_\omega \models \Sigma_{\text{UNA}}$ and by construction we also obtain $(a^{\mathcal{I}'_\omega}, b^{\mathcal{I}'_\omega}) \notin P^{\mathcal{I}'_\omega}$. Let $\varphi : \Delta^{\mathcal{I}'} \times \mathbb{N} \to \Delta$ be a bijection such that $\varphi(c^{\mathcal{I}'_\omega}) = c^{\mathcal{I}}$ for all $c \in N_I$. Such a φ exists, as $|\Delta^{\mathcal{I}'} \times \mathbb{N}| = |\Delta|$ and $\mathcal{I}'_\omega$ satisfies the unique name assumption. By Lemma 11, we get that $\varphi(\mathcal{I}'_\omega) \in \mathcal{M}(\Sigma)$. Moreover $(a^{\varphi(\mathcal{I}'_\omega)}, b^{\varphi(\mathcal{I}'_\omega)}) = (\varphi(a^{\mathcal{I}'_\omega}), \varphi(b^{\mathcal{I}'_\omega})) \notin P^{\varphi(\mathcal{I}'_\omega)}$. In particular,
$$(a^{\mathcal{I}}, b^{\mathcal{I}}) = (x,y) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J}}$$
which is a contradiction.
- Assume at least one of x, y is anonymous. W.l.o.g. let x be anonymous, the other case follows by symmetry. Considering $\mathcal{I}_\omega$, we again have $\mathcal{I}_\omega \models \Sigma_{\text{UNA}}$ by Lemma 9. By construction, $\langle x, 1\rangle$ is anonymous and $(\langle x,1\rangle, \langle y,0\rangle) \notin P^{\mathcal{I}_\omega}$. Let $\varphi : \Delta^{\mathcal{I}} \times \mathbb{N} \to \Delta$ be a bijection such that $\varphi(\langle x,1\rangle) = x$ and $\varphi(\langle y,0\rangle) = y$. Such a φ exists, since $|\Delta^{\mathcal{I}} \times \mathbb{N}| = |\Delta|$ and $\mathcal{I}_\omega$ satisfies the unique name assumption. By Lemma 11, we get that $\varphi(\mathcal{I}_\omega) \in \mathcal{M}(\Sigma)$. Moreover $(\varphi(\langle x,1\rangle), \varphi(\langle y,0\rangle)) \notin P^{\varphi(\mathcal{I}_\omega)}$. In particular,
$$(x,y) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J}}$$
which again is a contradiction.

Now we suppose that the first case does not hold. We have to show that $x = y$ and $\Sigma \models \top \sqsubseteq \exists P.\mathsf{Self}$. Again we assume to its contrary and make the following case distinction:

- $x \neq y$:
Now either both of x and y are named individuals but $\Sigma \not\models P(a,b)$ or at least one of them is anonymous. We can generate contradiction as above.
- $x = y$ and but $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists P.\mathsf{Self}$:
We have to distinguish two cases. First, suppose that x is a named individual i.e., there is $a \in N_I$ with $a^{\mathcal{I}} = x$. Now as $\Sigma_{\text{UNA}} \not\models P(a,a)$, this leads to contradiction as shown above.
Second, suppose that x is anonymous. Since every $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, it follows from Fact 6 that $\mathcal{J} \models \Sigma_{\text{UNA}}$ for every

$\mathcal{J} \in \mathcal{M}(\Sigma)$. This along with the fact that $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists P.\mathsf{Self}$ implies that there is some $\mathcal{I}' \in \mathcal{M}(\Sigma)$ such that $(u,u) \notin P^{\mathcal{I}'}$ for some $u \in \Delta$. We define a bijection $\varphi : \Delta \to \Delta$ such that $\varphi(u) = x$. By Lemma 11, we get that $\varphi(\mathcal{I}') \in \mathcal{M}(\Sigma)$. Moreover $(\varphi(u), \varphi(u)) \notin P^{\varphi(\mathcal{I}')}$. In particular,

$$(\varphi(u), \varphi(u)) = (x,x) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J}}$$

and therefore, by semantics, $(x,y) \notin \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ which is a contradiction. $\square$

Having established the above correspondences, we are able to define a translation procedure that maps (complex) epistemic concept expressions to non-epistemic ones which are equivalent in all models of Σ.

Definition 15. Given a $\mathcal{SRIQ}$ knowledge base Σ, we define the function Φ_Σ mapping $\mathcal{SROIQK}$ concept expressions to $\mathcal{SROIQ}$ concept expressions as follows (where we let $\{\} = \emptyset = \bot$)[4]:

$$\Phi_\Sigma : \begin{cases}
C \mapsto C \quad \text{if } C \text{ is an atomic or one-of concept, } \top \text{ or } \bot; \\
\mathbf{K}D \mapsto \begin{cases} \top & \text{if } \Sigma_{\text{UNA}} \models \Phi_\Sigma(D) \equiv \top \\ \{a \in N_I \mid \Sigma_{\text{UNA}} \models \Phi_\Sigma(D)(a)\} & \text{otherwise} \end{cases} \\
\exists \mathbf{K}S.\mathsf{Self} \mapsto \begin{cases} \exists S.\mathsf{Self} & \text{if } \Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \{a \in N_I \mid \Sigma_{\text{UNA}} \models S(a,a)\} & \text{otherwise} \end{cases} \\
C_1 \sqcap C_2 \mapsto \Phi_\Sigma(C_1) \sqcap \Phi_\Sigma(C_2) \\
C_1 \sqcup C_2 \mapsto \Phi_\Sigma(C_1) \sqcup \Phi_\Sigma(C_2) \\
\neg C \mapsto \neg \Phi_\Sigma(C) \\
\exists R.D \mapsto \exists R.\Phi_\Sigma(D) \quad \text{for non-epistemic role } R \\
\exists \mathbf{K}P.D \mapsto \bigsqcup_{a \in N_I} \{a\} \sqcap \exists P.(\{b \in N_I \mid \Sigma_{\text{UNA}} \models P(a,b)\} \sqcap \Phi_\Sigma(D)) \\
\qquad\qquad \sqcup \begin{cases} \Phi_\Sigma(D) & \text{if } \Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \\
\forall R.D \mapsto \forall R.\Phi_\Sigma(D) \quad \text{for non-epistemic role } R; \\
\forall \mathbf{K}P.D \mapsto \neg \Phi_\Sigma(\exists \mathbf{K}P.\neg D) \\
\geqslant nS.D \mapsto \geqslant nS.\Phi_\Sigma(D) \quad \text{for non-epistemic role } S; \\
\geqslant n\mathbf{K}S.D \mapsto \begin{cases} \bigsqcup_{a \in N_I} \{a\} \sqcap \geqslant nP.(\{b \in N_I \mid \Sigma_{\text{UNA}} \models P(a,b)\} \sqcap \Phi_\Sigma(D)) & \text{if } n > 1 \\ \Phi_\Sigma(\exists \mathbf{K}P.D) & \text{otherwise} \end{cases} \\
\leqslant nS.D \mapsto \leqslant nS.\Phi_\Sigma(D) \quad \text{for non-epistemic role } S; \\
\leqslant n\mathbf{K}S.D \mapsto \neg \Phi_\Sigma(\geqslant (n{+}1)\mathbf{K}S.D) \\
\Xi \mathbf{K}U.D \mapsto \Xi U.\Phi_\Sigma(D) \quad \text{for } \Xi \in \{\forall, \exists, \geqslant n, \leqslant n\}
\end{cases}$$

$\Diamond$

[4] W.l.o.g. we assume that in the definition of Φ_Σ, $n \geq 1$.

Now to see if this method is indeed correct, first in the following lemma, we show that the extension of a $\mathcal{SROIQK}$-concept and the extension of the $\mathcal{SROIQ}$-concept, obtained using the translation function Φ_Σ, agree under each model of the knowledge base.

Lemma 16. *Let Σ be a $\mathcal{SRIQ}$-knowledge base, x be an element of Δ, and C be a $\mathcal{SROIQK}$ concept. Then for any interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$, we have that $C^{\mathcal{I},\mathcal{M}(\Sigma)} = (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$.*

Proof. It suffices to show that for any $x \in \Delta$, $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ exactly when $x \in \Phi_\Sigma(C)^{\mathcal{I},\mathcal{M}(\Sigma)}$. To show this we use induction on the structure of the C. For the base case (C is an atomic concept) and the cases where $C = \top$ or $C = \bot$, the lemma follows immediately from the definition of Φ_Σ. For the cases, where $C = C_1 \sqcap C_2$, $C = C_1 \sqcup C_2$ or $C = \neg D$, it follows from the standard semantics and induction hypothesis. We focus on the rest of the cases in the following.

i. $C = \mathbf{K}D$ and $\Sigma_{\text{UNA}} \not\models D \equiv \top$:
By Lemma 12, $x \in (\mathbf{K}D)^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if there is an $a \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models D(a)$. This is equivalent to $x \in \{a \in N_I \mid \Sigma_{\text{UNA}} \models D(a)\}^{\mathcal{I},\mathcal{M}(\Sigma)}$ and hence, by definition of Φ_Σ, to $x \in [\Phi_\Sigma(\mathbf{K}D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

ii. $C = \mathbf{K}D$ and $\Sigma_{\text{UNA}} \models D \equiv \top$:
Note that it trivially holds that if $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ then $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $\Phi_\Sigma(C) = \top$. Hence we just prove that whenever $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ then $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ also. To contrary, suppose this is not the case i.e., $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ but $x \notin C^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, by definition, we get that

$$x \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

Therefore, there is an interpretation $\mathcal{I}' \in \mathcal{M}(\Sigma)$ such that $x \notin D^{\mathcal{I}'}$. Since $\mathcal{M}(\Sigma)$ is the epistemic model of Σ, hence $\mathcal{I}' \in \mathcal{M}(\Sigma)$ respects the unique name assumption and therefore, $\mathcal{I}' \models \Sigma_{\text{UNA}}$ with $D^{\mathcal{I}'} \neq \Delta$. Hence $\Sigma_{\text{UNA}} \not\models D \equiv \top$, which is a contradiction.

iii. $C = \exists \mathsf{K} S.\mathsf{Self}$
" $\Rightarrow$ "
We have to distinguish two cases.
First, we suppose that $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$, therefore by definition $\Phi_\Sigma(\exists \mathsf{K} S.\mathsf{Self}) = \exists S.\mathsf{Self}$. Now $x \in [\exists \mathsf{K} S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that for each $\mathcal{J} \in \mathcal{M}(\Sigma)$, we have that $(x, x) \in S^{\mathcal{J},\mathcal{M}(\Sigma)}$. In particular, $(x, x) \in S^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore, $x \in [\exists S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and hence

$x \in [\Phi_{\Sigma}(\exists \mathbf{K} S.\mathsf{Self})]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Second, suppose that $\Sigma_{\mathrm{UNA}} \not\models \top \sqsubseteq \exists S.\mathsf{Self}$. As $x \in [\exists \mathbf{K} S.\mathsf{Self}]$ implies that $(x, x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$, by Lemma 14 there is $a \in N_I$ such that $a^{\mathcal{I}} = x$ and $\Sigma_{\mathrm{UNA}} \models S(a, a)$ i.e., $a \in \{c \in N_I \mid \Sigma_{\mathrm{UNA}} \models S(c, c)\}$ which immediately implies that $x = a^{\mathcal{I}} \in [\Phi_{\Sigma}(\exists \mathbf{K} S.\mathsf{Self})]^{\mathcal{I},\mathcal{M}(\Sigma)}$ as per definition of Φ_{Σ}.

" $\Leftarrow$ "

Suppose that $\Phi_{\Sigma}(\exists \mathbf{K} S.\mathsf{Self}) = \exists \mathbf{K} S.\mathsf{Self}$. Hence it is the case that $\Sigma_{\mathrm{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$. Now as each model in $\mathcal{M}(\Sigma)$ satisfies UNA, by Fact 6, we have that $\mathcal{J} \models \Sigma_{\mathrm{UNA}}$ and hence $\mathcal{J} \models \top \sqsubseteq \exists S.\mathsf{Self}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$ i.e., for every $u \in \Delta$, we have that $(u, u) \in S^{\mathcal{J},\mathcal{M}(\Sigma)}$. In other words, for every $u \in \Delta$, we have that $(u, u) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. In particular, we have that $x \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore by semantics, $x \in [\exists \mathbf{K} S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Assume that $\Phi_{\Sigma}(\exists \mathbf{K} S.\mathsf{Self}) = \{c \in N_I \mid \Sigma_{\mathrm{UNA}} \models S(c, c)\}$. Consequently, there is $a \in N_I$ with $a^{\mathcal{I}} = x$ and $\Sigma_{\mathrm{UNA}} \models S(a, a)$ which by Lemma 14, implies that $(x, x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore, we get that $x \in [\exists \mathbf{K} S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

iv. $C = \exists P.D$ and P is a simple role:
By semantics, $x \in (\exists P.D)^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if there is $y \in \Delta$ such that $(x, y) \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $y \in [\Phi_{\Sigma}(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence it is equivalent to $x \in (\Phi_{\Sigma}(KD))^{\mathcal{I},\mathcal{M}(\Sigma)}$.

v. $C = \exists \mathbf{K} P.D$:
" $\Rightarrow$ "
$x \in [\exists \mathbf{K} P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that there is some $y \in \Delta$ with $(x, y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ such that $y \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $y \in [\Phi_{\Sigma}(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 14, $(x, y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that at least one of the following should hold.

- There are $a, b \in N_I$ with $a^{\mathcal{I}} = x$ and $b^{\mathcal{I}} = y$ such that $\Sigma_{\mathrm{UNA}} \models P(a, b)$: Consequently we have that $y = b^{\mathcal{I}} \in [\{c \in N_I \mid \Sigma_{\mathrm{UNA}} \models P(a, c)\} \sqcap \Phi_{\Sigma}(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Now as $\mathcal{M}(\Sigma)$ is an epistemic model, every interpretation in $\mathcal{M}(\Sigma)$ satisfies the UNA, and hence by Fact 6, for every $\mathcal{J} \in \mathcal{M}(\Sigma)$ we have that $\mathcal{J} \models \Sigma_{\mathrm{UNA}}$. This along with $\Sigma_{\mathrm{UNA}} \models P(a, b)$ implies that $(a^{\mathcal{I}}, b^{\mathcal{I}}) = (x, y) \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore, $x \in [\exists P.(\{c \in N_I \mid \Sigma_{\mathrm{UNA}} \models P(a, c)\} \sqcap \Phi_{\Sigma}(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, $x = a^{\mathcal{I}} \in [\{a\} \sqcap \{c \in N_I \mid \Sigma_{\mathrm{UNA}} \models P(a, c)\} \sqcap \Phi_{\Sigma}(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, which by definition of Φ_{Σ} implies that $x \in [\Phi_{\Sigma}(\exists \mathbf{K} P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
- $x = y$ and $\Sigma_{\mathrm{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$: As $y \in [\Phi_{\Sigma}(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore it immediately follows from the definition that $x \in [\Phi_{\Sigma}(\exists \mathbf{K} P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

" $\Leftarrow$ "

Suppose that $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. This means that at least one of the following should hold.

– $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$:
It implies that there is an $a \in N_I$ such that $a^\mathcal{I} = x$ and $a^\mathcal{I} \in [\exists P.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a,c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Consequently there is some $b \in N_I$ such that $b^\mathcal{I} \in [[\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a,c)\} \sqcap \Phi_\Sigma(D)]]^{\mathcal{I},\mathcal{M}(\Sigma)}$ i.e., $\Sigma_{\text{UNA}} \models P(a,b)$ and $b^\mathcal{I} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $b^\mathcal{I} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 14, $\Sigma_{\text{UNA}} \models P(a,b)$ implies that $(a^\mathcal{I}, b^\mathcal{I}) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore we get that $x = a^\mathcal{I} \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

– $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$:
Note that each $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, $\mathcal{J} \models \Sigma_{\text{UNA}}$. This implies that $\mathcal{J} \models \top \sqsubseteq \exists P.\mathsf{Self}$. In other words, for every $u \in \Delta$, we have that $(u,u) \in P^\mathcal{J}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$ and therefore, by semantics, we get that $(u,u) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. In particular, $(x,x) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. Now as $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, we get that $x \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

vi. $C = \geqslant n\mathbf{K}S.D$:
"$\Rightarrow$"
Depending on n we distinguish the following cases.

– $n = 1$:
$x \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ means that $x \in [\exists \mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Earlier we showed that this is the case iff $x \in [\Phi_\Sigma(\exists \mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore by definition, $x \in [\Phi_\Sigma(\geqslant 1\mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

– $n > 1$:
$x \in [\geqslant n\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that there are $y_1, \ldots, y_m$ with $m \geq n$ such that $(x, y_i) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y_i \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. By induction, $y_i \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. By Lemma 14, we have $a, b_1, \ldots, b_n \in N_I$ such that $a^\mathcal{I} = x$, $b_i^\mathcal{I} = y_i$ and $\Sigma_{\text{UNA}} \models S(a, b_i)$ for each $i \leq m$. Now as $m \geq n$ and $b_i^\mathcal{I} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for $i \leq m$, it follows from the semantics that $x = a^\mathcal{I} \in [\geqslant nS.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models S(a,c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, using definition of Φ_Σ, we obtain that $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $x \in \{a\}^{\mathcal{I},\mathcal{M}(\Sigma)}$.

"$\Leftarrow$"
Suppose that $n > 1$. Therefore,

$$\Phi_\Sigma(\geqslant n\mathbf{K}S.D) = \bigsqcup_{c \in N_I} \{c\} \sqcap \geqslant nS.(\{c' \in N_I \mid \Sigma_{\text{UNA}} \models S(c,c')\} \sqcap \Phi_\Sigma(D))$$

Now $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that there are $a, b_1, \ldots, b_m \in N_I$, for $m \geq n$, such $a^\mathcal{I} = x$, $\Sigma_{\text{UNA}} \models S(a, b_i)$ and $b_i^\mathcal{I} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$

for each $i \leq m$. Since each $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, $\mathcal{J} \models \Sigma_{\text{UNA}}$ and hence we get that $(a^{\mathcal{J}}, b_i^{\mathcal{J}}) \in S^{\mathcal{J}}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$. Hence, It follows from the semantics that $(a^{\mathcal{I}}, b_i^{\mathcal{I}}) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. Now as $m \geq n$ and $b_i^{\mathcal{I}} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$(by induction), we get that $x \in [\geqslant n\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Now assume that $n = 1$. Hence, $\Phi_\Sigma(\geqslant n\mathbf{K}S.D) = \Phi_\Sigma(\exists\mathbf{K}S.D)$. Now for $x \in [\exists\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ at least one of the following holds:

- there is $a, b \in N_I$ with $a^{\mathcal{I}} = x$ such that $\Sigma_{\text{UNA}} \models S(a,b)$ and $b^{\mathcal{I}} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $b^{\mathcal{I}} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 14, we get that $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ which along with $b^{\mathcal{I}} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that $x = a^{\mathcal{I}} \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
- $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$. By Lemma 14, we get that $(x,x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. By induction we have that $x \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ which immediately implies that $x \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

vii. The rest of the cases can be proved in a similar fashion.

Moreover Lemma 16 allows to establish the result that the translation function Φ_Σ can be used to reduces the problem of entailment of $\mathcal{SROIQK}$ axioms by $\mathcal{SRIQ}$ knowledge bases to the problem of entailment of $\mathcal{SROIQ}$ axioms, formally put into the following theorem.

Theorem 17. *For a $\mathcal{SRIQ}$ knowledge base Σ, $\mathcal{SROIQK}$-concepts C and D and an individual a the following hold:*

1. $\Sigma \Vdash C(a)$ *exactly if* $\Sigma_{\text{UNA}} \models \Phi_\Sigma(C)(a)$.
2. $\Sigma \Vdash C \sqsubseteq D$ *exactly if* $\Sigma_{\text{UNA}} \models \Phi_\Sigma(C) \sqsubseteq \Phi_\Sigma(D)$.

Proof. For the first case, we see that $\Sigma \Vdash C(a)$ is equivalent to $a^{\mathcal{I},\mathcal{M}(\Sigma)} \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ which by Lemma 16 is the case exactly if $a^{\mathcal{I},\mathcal{M}(\Sigma)} \in \Phi_\Sigma(C)^{\mathcal{I},\mathcal{M}(\Sigma)}$ for all $\mathcal{I} \in \mathcal{M}(\Sigma)$. Since $\Phi_\Sigma(C)$ does not contain any **K**s, this is equivalent to $a^{\mathcal{I}} \in \Phi_\Sigma(C)^{\mathcal{I}}$ and hence to $\mathcal{I} \models \Phi_\Sigma(C)(a)$ for all $\mathcal{I} \in \mathcal{M}(\Sigma)$. Now we can invoke Fact 6 and Lemma 9 to see that this is the case if and only if $\Sigma_{\text{UNA}} \models \Phi_\Sigma(C)(a)$. The second case is proven in exactly the same fashion. □

Hence standard DL-reasoners can be used in order to answer epistemic queries. It can be seen from the definition of Φ_Σ that deciding epistemic entailment along those lines may require deciding many classical entailment problems and hence involve many calls to the reasoner. Nevertheless, the number of reasoner calls is bounded by the number of **K**s occurring in the query.

5 Semantical Problems Caused by Nominals and the Universal Role

One of the basic assumptions that is made regarding the epistemic interpretations is the *common domain assumption* as mentioned in Section 3. It basically has two parts: all the interpretations considered in an epistemic interpretation share the same fixed domain and the domain is infinite. However, there is no prima facie reason, why the domain that is described by a knowledge base should not be finite, yet finite models are excluded from the consideration entirely. We have shown that this is still tolerable for description logics up to $\mathcal{SRIQ}$ due to the fact that every finite model of a knowledge base gives rise to an infinite one that behaves the same (i.e. the two models cannot be distinguished by means of the underlying logic), as shown in Lemma 9. However, this situation changes once nominals or the universal role are allowed. In fact, the axioms $\top \sqsubseteq \{a, b, c\}$ or $\top \sqsubseteq \leqslant 3U.\top$ have only models with at most three elements. Consequently, according to the prevailing epistemic semantics, these axioms are epistemically unsatisfiable. In general, the coincidence of $\Vdash$ and $\models$ under the UNA which holds for nonepistemic KBs and axioms up to $\mathcal{SRIQ}$ does not hold any more, once nominals or the universal role come into play.

We believe that this phenomenon is not intended but rather a side effect of a semantics crafted for and probed against less expressive description logics, as it contradicts the intuition behind the **K** operator. A refinement of the semantics in order to ensure an intuitive behavior also in the presence of very expressive modeling features is subject of ongoing research.

6 A System

To check the feasibility of our method in practice, we have implemented a system that we called *EQuIKa*[5] and performed some first experiments for epistemic querying.

Implementation The *EQuIKa* system implements the transformation Φ of an epistemic concept to its non-epistemic version from Definition 15 involving calls to an underlying standard DL reasoner that offers the reasoning task of instance retrieval. To obtain an efficient implementation of Φ it is crucial to keep the number of calls to the DL reasoner minimal.

[5] **E**pistemic **Qu**erying **I**nterfance **Ka**rlsruhe.

Algorithm 1 translate (Σ, C) – Translate epistemic query concepts to non-epistemic ones

Require: a $\mathcal{SRIQ}$ knowledge base Σ, an epistemic concept C
Ensure: the return value is the non-epistemic concept $\Phi(C)$

translate $(\Sigma, C = \mathbf{K}D)$
  $\mathcal{X}$:= retrieveInstances $(\Sigma$, translate $(\Sigma, D))$
  return $\{\dots, o_i, \dots\} \quad , o_i \in \mathcal{X}$
translate $(\Sigma, C = \exists\mathbf{K}R.D)$
  $\mathcal{X}_D$:= retrieveInstances $(\Sigma$, translate $(\Sigma, D))$
  $\mathcal{X}$:= retrieveInstances $(\Sigma, \exists R.\{\dots, o_i, \dots\}) \quad , o_i \in \mathcal{X}_D$
  return $\{\dots, o_i, \dots\} \quad , o_i \in \mathcal{X}$
translate $(\Sigma, C = \forall\mathbf{K}R.D)$
  $\mathcal{X}_{\bar{D}}$:= retrieveInstances $(\Sigma$, translate $(\Sigma, \neg D))$
  $\mathcal{X}$:= retrieveInstances $(\Sigma, \exists R.\{\dots, o_i, \dots\}) \quad , o_i \in \mathcal{X}_{\bar{D}}$
  return $\neg\{\dots, o_i, \dots\} \quad , o_i \in \mathcal{X}$
translate $(\Sigma, C = \geq n\mathbf{K}R.D)$
  $\mathcal{X}_D$:= retrieveInstances $(\Sigma$, translate $(\Sigma, D))$
  $\mathcal{X}$:= retrieveInstances $(\Sigma, \geq nR.\{\dots, o_i, \dots\}) \quad , o_i \in \mathcal{X}_D$
  return $\{\dots, o_i, \dots\} \quad , o_i \in \mathcal{X}$
translate $(\Sigma, C = \leq n\mathbf{K}R.D)$
  $\mathcal{X}_D$:= retrieveInstances $(\Sigma$, translate $(\Sigma, D))$
  $\mathcal{X}$:= retrieveInstances $(\Sigma, \geq (n+1)R.\{\dots, o_i, \dots\}) \quad , o_i \in \mathcal{X}_D$
  return $\neg\{\dots, o_i, \dots\} \quad , o_i \in \mathcal{X}$
translate $(\Sigma, C = \dots)$
  ...

With Algorithm 1 we provide such an efficient implementation, exploiting the fact that extensions of epistemic roles (that occur in role restrictions) only contain known individuals. It shows the transformation in terms of virtual recursive translation functions for the various cases of epistemic concept expressions[6].

From Algorithm 1, it can be seen that the number of calls to the underlying DL reasoner is at most twice the number of **K**-operators that occur in the original query. This is much better than a naive implementation of Φ according to Definition 15 with iteration over intermediate retrieved individuals.

The *EQuIKa* system is implemented on top of the OWL-API[7] extending its classes and interfaces with constructs for epistemic concepts and roles, as shown by the UML class diagram in Figure 1. The new

[6] The less interesting cases of non-epistemic concept expressions are not exposed, as their implementation trivially follows Definition 15.
[7] http://owlapi.sourceforge.net/

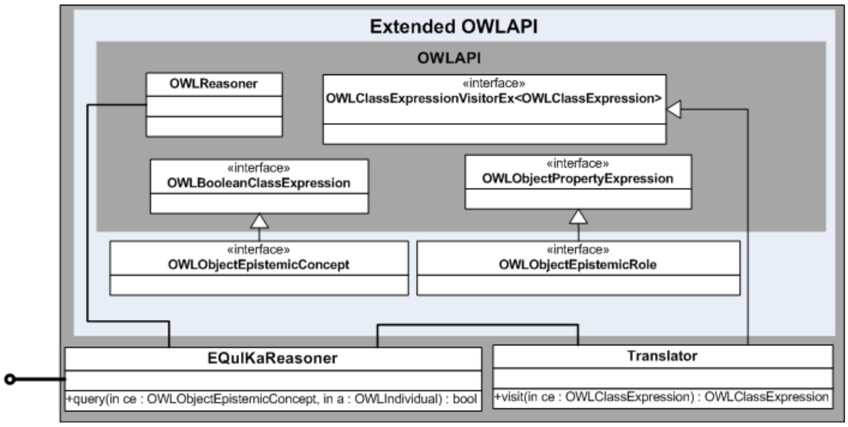

Fig. 1. The *EQuIKa*-system extending the OWL-API

Table 2. Concepts used for instance retrieval experiments.

C_1	$\exists hasWineDescriptor.WineDescriptor$
EC_1	$\exists \mathbf{K} hasWineDescriptor.\mathbf{K}\, WineDescriptor$
C_2	$\forall hasWineDescriptor.WineDescriptor$
EC_2	$\forall \mathbf{K} hasWineDescriptor.\mathbf{K}\, WineDescriptor$
C_3	$\exists hasWineDescriptor.WineDescriptor \sqcap \exists madeFromFruit.WineGrape$
EC_3	$\exists \mathbf{K} hasWineDescriptor.\mathbf{K}\, WineDescriptor \sqcap \exists \mathbf{K} madeFromFruit.\mathbf{K}\, WineGrape$
C_4	$WhiteWine \sqcap \neg\exists locatedIn.\{FrenchRegion\}$
EC_4	$\mathbf{K}\, WhiteWine \sqcap \neg\exists \mathbf{K} locatedIn.\{FrenchRegion\}$
C_5	$Wine \sqcap \neg\exists hasSugar.\{Dry\} \sqcap \neg\exists hasSugar.\{OffDry\} \sqcap \neg\exists hasSugar.\{Sweet\}$
EC_5	$\mathbf{K}\, Wine \sqcap \neg\exists \mathbf{K} hasSugar.\{Dry\} \sqcap \neg\exists \mathbf{K} hasSugar.\{OffDry\} \sqcap \neg\mathbf{K}\exists hasSugar.\{Sweet\}$

types OWLObjectEpistemicConcept and OWLObjectEpistemicRole are derived from the respective standard types OWLBooleanClassExpression and OWLObjectPropertyExpression to fit the design of the OWL-API.

Using these types, the transformation Φ is implemented in the class Translator following the visitor pattern mechanism built in the OWL-API, which is indicated by the virtual translation functions with different arguments in Algorithm 1. Finally, the EQuIKaReasoner uses both a Translator together with an OWLReasoner to perform epistemic reasoning tasks.

Experiments For the purpose of testing, we consider two versions of the wine ontology[8] which we call Wine1 and Wine2 each with number of instances 483 and 1127 respectively. As a measure for our test, we consider the time required to compute the instances of a concept. This suffices as

[8] http://www.w3.org/TR/owl-guide/wine.rdf

Table 3. Evaluation

Ontology	Concept	$\mathsf{t}_{(C_i)}$	$\|C_i\|$	Concept	$\mathsf{t}_{\mathsf{T}(EC_i)}$	$\mathsf{t}_{(EC_i)}$	$\|EC_i\|$
Wine 1	C_1	2.13	159	EC_1	46.98	0.04	3
	C_2	0.01	483	EC_2	0.18	0.00	0
	C_3	28.90	159	EC_3	79.43	6.52	3
	C_4	0.13	0	EC_4	95.60	107.82	72
	C_5	52.23	80	EC_5	60.78	330.49	119
Wine 2	C_1	8.51	371	EC_1	351.78	0.13	308
	C_2	0.30	1127	EC_2	0.127	0.00	0
	C_3	227.10	371	EC_3	641.24	19.58	7
	C_4	0.34	0	EC_4	865.04	840.97	168
	C_5	295.87	240	EC_5	381.41	2417.65	331

entailment check can not be harder than instance retrieval. We consider different epistemic concepts. For each concept C of these concepts, we consider a non-epistemic concept obtained from C by dropping the $\mathbf{K}$-operators occuring in it, which are given as in Table 2. Given a concept C, by $\mathsf{t}_{(C)}$ we represent the time in seconds required to compute the instances of the concept C. Similary $|C_i|$ represent the number of instances computed. Finally for an epistemic concept EC_i, $\mathsf{t}_{\mathsf{T}(EC_i)}$ represents the time taken to translate it to its non-epistemic equivalent. Table 3 provides our evaluation results for every ontology and every concept under consideration.

One can see from the evaluation results in Table 3 that the time required to compute the number of instances is feasible; it is roughly in the same order of magnitude as for non-epistemic concepts. Note also that the runtime comparison between epistemic concepts EC_i and their non-epistemic counterparts t_{C_i} should be taken with a grain of salt as they are semantically different in general, as also indicated by the fact that there are cases where retrieval for the epistemic concept takes less time than for the non-epistemic version. As a general observation, we noticed that instances retrieval for an epistemic concept where a $\mathbf{K}$-operator occurs within the scope of a negation, tends to require much time.

7 Conclusion

We have provided a way to answer epistemic queries to restricted OWL 2 DL ontologies via a reduction to a series of standard reasoning steps. This enables the deployment of today's highly optimized OWL inference engines for this non-standard type of queries. Experiments have shown that the approach is computationally feasible with runtimes in the same order of magnitude as standard (non-epistemic) reasoning tasks.

We identify the following avenues for future research: first and foremost we want to extend the expressivity of the underlying knowledge base to full OWL 2 DL, including nominals and the universal role. To this end, we have to alter the semantics and relinquishing the common domain assumption, to retain an intuitive entailment behavior. Second, we will provide a language extension to OWL 2 for epistemic operators in order to provide for a coherent way of serializing epistemic axioms. Finally we will investigate to which extent the promoted blackbox approach can be extended to the case where the epistemic operator occurs inside the considered knowledge base – note however, that in this case there is no unique epistemic model anymore.

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