# Uniform Interpolation in General $\mathcal{EL}$ Terminologies

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**Abstract.** Recently, different Forgetting approaches for knowledge bases expressed in different logics were proposed. It was shown, that the result may not exist in the presence of terminological cycles and sufficient, but not necessary conditions for its existence in $\mathcal{EL}$ were proposed. In this paper, we show that a uniform interpolant of any $\mathcal{EL}$ terminology w.r.t. any signature always exists in $\mathcal{EL}$ enriched with least and greatest fixpoint constructors and show how it can be computed by reducing the problem to the computation of Most General Subconcepts and Most Specific Superconcepts for atomic concepts. Moreover, we give the exact conditions for the existence of a uniform interpolant in $\mathcal{EL}$ and show how it can be obtained using our algorithms.

## 1 Introduction

The importance of non-standard reasoning services supporting knowledge engineers in modelling a particular domain or in understanding existing models by visualizing implicit dependencies between concepts and roles was pointed out by the research community [3], [5]. An example of such reasoning services supporting knowledge engineers in different activities is the uniform interpolation. In particular for the understanding and the development of complex knowledge bases, e.g., those consisting of *general concept inclusions* (GCIs), the appropriate tool support would be beneficial. However, the existing approaches to uniform interpolation and some structurally similar reasoning problems either do not consider GCIs, since the result may not exist in the presence of terminological cycles, or rely on sufficient but not necessary termination conditions. E.g., decomposing $\mathcal{EL}$ knowledge bases into logically independent modules [8] is restricted to role-acyclic $\mathcal{EL}$ TBoxes; uniform interpolation in a Horn extension of $\mathcal{EL}$[9] is based on acyclicity conditions; Colucci et al.[5] present a framework for non-standard reasoning services based on Tableau extended with variable substitution by modelling the problems as second-order concept expressions. However, termination and decidability of the satisfiability of the corresponding formulas remained open. Wang et al.[13] propose an approach to uniform interpolation in $\mathcal{ALC}$ w.r.t. general terminologies by encoding $\mathcal{ALC}$ TBoxes as concepts, which is not applicable in case of $\mathcal{EL}$. Currently, the exact conditions for the existence of uniform interpolation in $\mathcal{EL}$ remain undetermined.

Clearly, the existence of the results for such reasoning problems is closely related to the notion of fixpoint semantics. For instance, Baader [2] shows that the structurally similar problems of computing Least Common Subsumer and Most Specific Concept can always be solved in cyclic classical TBoxes w.r.t. to greatest fixpoint semantics.

Similar results were obtained for general $\mathcal{EL}$ TBoxes with descriptive semantics[11] , however extended with the greatest fixpoint constructor ($\mathcal{EL}_\nu$). In this paper, we extend the above results by showing that uniform interpolants preserving all $\mathcal{EL}$ consequences of general $\mathcal{EL}$ terminologies w.r.t. an arbitrary signature can always be expressed in an extension of $\mathcal{EL}$ with least fixpoint and greatest fixpoint constructors $\mu, \nu$ as well as the disjunction used only on the left-hand side of concept inclusions. We propose the algorithms for computing such uniform interpolants based on the notion of *most general subconcepts* and *most specific superconcepts*.

In the usual application scenarios it is rather useful to obtain uniform interpolants expressed in the DL of the original terminology instead of introducing additional language constructs. Therefore, in addition to the above algorithms, we derive the existence criteria for uniform interpolants in $\mathcal{EL}$ (i.e., expressed without the above extension) and show how such a uniform interpolant can be obtained using our algorithms.

## 2 Preliminaries

Let $N_C$ and $N_R$ be countably infinite and mutually disjoint sets of concept symbols and role symbols. An $\mathcal{EL}$ concept $C$ is defined as

$$C ::= A|\top|C \sqcap C|\exists r.C$$

where $A$ and $r$ range over $N_C$ and $N_R$, respectively. In the following, we use symbols $A, B$ to denote atomic concepts and $C, D$ to denote arbitrary concepts. A *terminology* or *TBox* consists of *concept inclusion* axioms $C \sqsubseteq D$ and *concept equivalence* axioms $C \equiv D$ used as a shorthand for $C \sqsubseteq D$ and $D \sqsubseteq C$. While knowledge bases in general can also include a specification of individuals with the corresponding concept and role assertions (ABox), in this paper we abstract from ABoxes and concentrate on TBoxes. The signature of an $\mathcal{EL}$ concept $C$ or an axiom $\alpha$, denoted by $\mathrm{sig}(C)$ or $\mathrm{sig}(\alpha)$, respectively, is the set of concepts and role symbols occurring in it. The signature of a TBox $\mathcal{T}$, in symbols $\mathrm{sig}(\mathcal{T})$, is analogously $N_C \cup N_R$. In what follows, we denote the set $N_C \cup \{\top\}$ as $N_C^+$.

Before introducing the fixpoint operators, we recall the semantics of the above introduced DL constructs, which is defined by the means of interpretations. An interpretation $\mathcal{I}$ is given by the domain $\Delta^\mathcal{I}$ and a function $\cdot^\mathcal{I}$ assigning each concept $A \in N_C$ a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and each role $r \in N_R$ a subset $r^\mathcal{I}$ of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. The interpretation of $\top$ is fixed to $\Delta^\mathcal{I}$. The interpretation of an arbitrary $\mathcal{EL}$ concept is defined inductively, i.e., $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$ and $(\exists r.C)^\mathcal{I} = \{x \mid (x, y) \in r^\mathcal{I}, y \in C^\mathcal{I}\}$. An interpretation $\mathcal{I}$ satisfies an axiom $C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$. $\mathcal{I}$ is a model of a TBox, if it satisfies all of its axioms. We say that a TBox $\mathcal{T}$ entails an axiom $\alpha$, if $\alpha$ is satisfied by all models of $\mathcal{T}$. In combination with fixpoint constructors, we will additionally use *concept disjunction* $C \sqcup D$, the semantics of which is defined by $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$.

We now introduce the logics $\mathcal{EL}_{\mu(\sqcup),\nu}$, a fragment of monadic second order logics that we use to compute uniform interpolants of general $\mathcal{EL}$ TBoxes. $\mathcal{EL}_{\mu(\sqcup),\nu}$ is an extension of $\mathcal{EL}$ by the two fixpoint constructors, $\nu X.C_\nu$ [11] and $\mu X.C_\mu$ [4]. $X$ is an element of the countably infinite set of concept variables $N_V$ and $C_\nu$, $C_\mu$ are constructed as follows:

$$C_\nu ::= X|A|\top|\nu X.C_\nu|C_\nu \sqcap C_\nu|\exists r.C_\nu$$

$$C_\mu ::= X|A|\top|\mu X.C_\mu|C_\mu \sqcup C_\mu|C_\mu \sqcap C_\mu|\exists r.C_\mu$$

where $A$ ranges over atomic concepts and $X$ ranges over $N_V$. All $\mathcal{EL}_\nu$ concepts and all $\mathcal{EL}_{\mu(\sqcup)}$ concepts are closed $C_\nu$ and $C_\mu$ expression, i.e., all concept variables are bound by the corresponding fixpoint constructor. Note that we define $\mathcal{EL}_\nu$ concepts and all $\mathcal{EL}_{\mu(\sqcup)}$ concepts in such a way, that no concept can contain both fixpoint constructors, i.e., we do not combine the two constructors within concepts. The semantics of the fixpoint constructors is defined using a mapping $\vartheta$ of concept variables to subsets of $\Delta^\mathcal{I}$. For an $\mathcal{EL}_{\mu(\sqcup),\nu}$ concept $C$ and $W \subseteq \Delta^\mathcal{I}$, we denote a replacement of $X$ by $W$ as $C^{\mathcal{I},\vartheta[X\rightarrow W]}$. The semantics of $\mathcal{EL}_{\mu(\sqcup),\nu}$ concepts is defined by

$$(\nu X.C)^{\mathcal{I},\vartheta} = \bigcup\{W \subseteq \Delta^\mathcal{I}|W \subseteq C^{\mathcal{I},\vartheta[X\rightarrow W]}\}$$

$$(\mu X.C)^{\mathcal{I},\vartheta} = \bigcap\{W \subseteq \Delta^\mathcal{I}|C^{\mathcal{I},\vartheta[X\rightarrow W]} \subseteq W\}.$$

In order to allow for more succinct concept expressions, we use an extended version of the fixpoint constructs allowing for mutual recursion [12], [11]. The extended constructors have the form $\nu_i X_1...X_n.C_{\nu,1},...,C_{\nu,n}$ and $\mu_i X_1...X_n.C_{\mu,1},...,C_{\mu,n}$ with $1 \leq i \leq n$. The semantics is defined as

$$(\nu_i X_1...X_n.C_1,...,C_n)^{\mathcal{I},\vartheta} = \bigcup\{W_i\}$$

$$(\mu_i X_1...X_n.C_1,...,C_n)^{\mathcal{I},\vartheta} = \bigcap\{W_i\}$$

such that there are $W_1,...,W_{i-1},W_{i+1},...,W_n$ with respectively $W_j \subseteq C_j^{\mathcal{I},\vartheta[X_1\rightarrow W_1,...,X_n\rightarrow W_n]}$ and $C_j^{\mathcal{I},\vartheta[X_1\rightarrow W_1,...,X_n\rightarrow W_n]} \subseteq W_j$ for $1 \leq j \leq n$.

## 3 TBox Inseparability and Uniform Interpolation

Intuitively, two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are inseparable w.r.t. a signature $\Sigma$ if they have the same $\Sigma$ consequences, i.e., consequences whose signature is a subset of $\Sigma$. Depending on the particular application requirements, the expressivity of those $\Sigma$ consequences can vary from subsumption queries and instance queries to conjunctive queries. In this paper, we investigate forgetting based on concept inseparability of general $\mathcal{EL}$ terminologies defined analogously to previous work on inseparability, e.g., [10] or [9], as follows:

**Definition 1.** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two general $\mathcal{EL}$ TBoxes and $\Sigma$ a signature. $\mathcal{T}_1$ and $\mathcal{T}_2$ are concept-inseparable w.r.t. $\Sigma$, in symbols $\mathcal{T}_1 \equiv^c_\Sigma \mathcal{T}_2$, if for all $\mathcal{EL}$ concepts $C, D$ with $sig(C) \cup sig(D) \subseteq \Sigma$ holds $\mathcal{T}_1 \models C \sqsubseteq D$, iff $\mathcal{T}_2 \models C \sqsubseteq D$.*

Given a signature $\Sigma$ and a TBox $\mathcal{T}$, the aim of uniform interpolation or forgetting is to determine a TBox $\mathcal{T}'$ with sig$(\mathcal{T}') \subseteq \Sigma$ such that $\mathcal{T} \equiv^c_\Sigma \mathcal{T}'$. $\mathcal{T}'$ is also called a Uniform Interpolant (UI) of $\mathcal{T}$ w.r.t. $\Sigma$. As demonstrated by the following example, in the presence of cyclic concept inclusions, a UI might not exist for a particular $\mathcal{T}$ and a particular $\Sigma$, i.e., it might be not expressible as a finite set of finite axioms using only the language constructs of $\mathcal{EL}$ and the signature $\Sigma$.

*Example 1.* Forgetting the concept $A$ in the TBox $\mathcal{T} = \{A' \sqsubseteq A, A \sqsubseteq A'', A \sqsubseteq \exists r.A, \exists s.A \sqsubseteq A\}$ results in an infinite chain of consequences $A' \sqsubseteq \exists r.\exists r.\exists r....A''$ and $\exists s.\exists s.\exists s....A' \sqsubseteq A''$ containing nested existential quantifiers of unbounded depth.

Clearly, if the TBox in the above example is interpreted w.r.t. descriptive semantics, no most specific superconcept of $A'$ exists, while it can be easily expressed using the greatest fixpoint constructor $\nu$ thereby resulting in an inclusion axiom $A' \sqsubseteq \nu X.(A'' \sqcap \exists r.X)$. The most general subconcept of $A''$ can be expressed accordingly by the means of the least fixpoint constructor $\mu$, i.e., $\mu X.(A' \sqcup \exists s.X) \sqsubseteq A''$. In the following, we show that the corresponding UI of $\mathcal{T}$ w.r.t. $\Sigma$ for any $\mathcal{EL}$ TBox $\mathcal{T}$ and any signature $\Sigma$ can always be expressed in $\mathcal{EL}_{\mu(\sqcup),\nu}$. For this purpose, we reduce the problem of computing UI to the problem of computing *most general subconcepts* $\mathtt{MGS}(\Sigma, \mathcal{T}, A)$ and *most specific superconcepts* $\mathtt{MSS}(\Sigma, \mathcal{T}, A)$ for each concept $A \in \text{sig}(\mathcal{T})$.

**Definition 2.** *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and $\Sigma$ a signature. Further, let $A \in N_C$ and $C, C_i$ a set of $\mathcal{EL}$ concepts. $C = \mathtt{MSS}(\mathcal{T}, \Sigma, A)$ if the following conditions are fulfilled:*

- *$sig(C) \subseteq \Sigma$,*
- *for all $\Sigma$ concepts $D$ holds $\mathcal{T} \models A \sqsubseteq D$ if $\mathcal{T} \models C \sqsubseteq D$;*

$\bigsqcup_{1 \leq i \leq n} C_i = \mathtt{MGS}(\mathcal{T}, \Sigma, A)$ *if the following conditions are fulfilled:*

- *$sig(C_i) \subseteq \Sigma$,*
- *for all $\Sigma$ concepts $D$ holds $\mathcal{T} \models D \sqsubseteq A$ if $\mathcal{T} \models D \sqsubseteq \bigsqcup_{1 \leq i \leq n} C_i$.*

Note that, if $\mathtt{MGS}(\mathcal{T}, \Sigma, A)$ consists of several incomparable disjuncts $C_i$, it cannot be expressed by an $\mathcal{EL}$ concept. In the following, it will come into notice that this is not further problematic for the computation of UI, since the disjunction appears only on the left-hand side and can therefore be expressed by the means of several inclusion axioms. Analogously to MGS, we consider MSS as a conjunction using the notation $\mathtt{SUP}(\mathcal{T}, \Sigma, A) = \{C_i | \mathtt{MSS}(\mathcal{T}, \Sigma, A) = \bigsqcap_{1 \leq i \leq n} C_i\}$. The corresponding notation for disjuncts $C_i$ within MGS is $\mathtt{SUB}(\mathcal{T}, \Sigma, A)$. If the TBox $\mathcal{T}$ and the signature $\Sigma$ do not change, we omit them and simply write $\mathtt{MSS}(A), \mathtt{MGS}(A), \mathtt{SUP}(A)$ and $\mathtt{SUB}(A)$. For the remainder of this paper, we fix $\mathcal{T}$ to be a general $\mathcal{EL}$ TBox and $\Sigma$ a signature. Assuming that the TBox is normalized as described in the next Section, we compute a UI given $\mathtt{SUB}(A)$ and $\mathtt{SUP}(A)$ for each $A \in N_C$ as follows:

**Definition 3.** $\mathtt{UI}(\mathcal{T}, \Sigma) = \bigcup_{1 \leq i \leq 3} M_i$ *with*

- $M_1 = \{A \sqsubseteq D | A \in N_C \cap \Sigma, D \in \mathtt{SUP}(A)\}$
- $M_2 = \{C \sqsubseteq A | A \in N_C \cap \Sigma, C \in \mathtt{SUB}(A)\}$
- $M_3 = \{C \sqsubseteq D|$ *there is* $A \in N_C \cap \overline{\Sigma}$, *such that* $C \in \mathtt{SUB}(A)$ *and* $D \in \mathtt{SUP}(A)\}$

If $\mathtt{SUB}(A)$ and $\mathtt{SUP}(A)$ can be uniquely determined for a particular TBox $\mathcal{T}$ and signature $\Sigma$, the TBox $\mathtt{UI}(\mathcal{T}, \Sigma)$ is also uniquely determined. After introducing the normalization and the formal properties of SUP and SUB, we will prove that $\mathtt{UI}(\mathcal{T}, \Sigma) \equiv^c_\Sigma \mathcal{T}$.

## 4 Normalization

In order to simplify the computation of SUB and SUP, we apply the following normalization thereby restricting the syntactic form of $\mathcal{T}$. Analogously to the normalization employed in other approaches ([1], [7], [9]), we decompose complex axioms into syntactically simple ones. The decomposition is realized recursively by replacing expressions $B_1 \sqcap ... \sqcap B_n$ and $\exists r.B$ with fresh concept symbols until all axioms in $\mathcal{T}$ have the form:

- $A \sqsubseteq B$
- $A \equiv B_1 \sqcap ... \sqcap B_n$
- $A \equiv \exists r.B$

where $A, B, B_i \in N_C \cup \{\top\}$ and $r \in N_R$. For this purpose, we introduce a minimal required set of fresh concept symbols $A' \in N_D$ and the corresponding definition axioms ($A' \equiv C$) for each of them. In what follows, we assume that knowledge bases are normalized and refer to $N_C \cup N_D$ as $N_C$. Since concept symbols in $N_D$ are fresh, they do not appear in $\Sigma$ and are therefore elements of the forgotten signature $\overline{\Sigma}$. Further, we assume that equivalent concept symbols have been replaced by a single representative of the corresponding equivalence class.[1] The following lemma summarizes the properties of normalized TBoxes.

**Lemma 1.** *Any $\mathcal{T}$ can be extended into a normalized TBox $\mathcal{T}'$ and each model of $\mathcal{T}$ can be extended into a model of $\mathcal{T}'$.*

*Proof.* All introduced concepts in $N_D$ are defined in terms of concepts with $\text{sig}(C) \subseteq \text{sig}(\mathcal{T})$, therefore each model of $\mathcal{T}$ can be extended into a model of $\mathcal{T}'$.

## 5 Computing SUB and SUP for Acyclic Terminologies

Given an acyclic $\mathcal{EL}$ TBox $\mathcal{T}$ and a signature $\Sigma$, Algorithms 1 and 2 compute for each $A \in N_C$ the elements of SUB($A$) and SUP($A$), respectively. The indirectly recursive computation is derived from a Gentzen-style proof system relying on the above specified normal form. Both algorithms proceed along the definitions for $A$ in $\mathcal{T}$ as well as the inclusions between atomic concepts entailed by $\mathcal{T}$. Depending on whether or not a concept $B$ referenced in those definitions and inclusion axioms is in $\Sigma$, the procedure $\text{SUB}_F(B, A)$ ($\text{SUP}_F(B, A)$) returns $B$ itself, which is the basecase of the computation, or calls $\text{SUB}_S(B)$ ($\text{SUP}_S(B)$). The second parameter of $\text{SUB}_F$ is not relevant in case of acyclic TBoxes, but it will become important for computations based on fixpoint constructs. The functions REDUCE and $\text{REDUCE}_C$ eliminate redundancy within the computed results, which is not just an optimization, but will also play an important role in proofs within the last section. The first of the two functions expects as input a set of concepts and returns a subset of this set containing only incomparable concepts. The second function accepts a conjunction and returns a conjunction consisting only of incomparable

[1] The elimination of equivalent symbols does not affect the correctness or completeness of the uniform interpolation, since the removed symbols can easily be included into the resulting TBox.

**Algorithm 1** computing $\mathtt{SUB}_S(A)$ for an $\mathcal{EL}$ TBox $\mathcal{T}$ and a signature $\Sigma$

$\mathtt{SUB} \leftarrow \bigcup \mathtt{SUB}_F(D, A), D \in N_C$ such that $\mathcal{T} \models D \sqsubseteq A$
**for all** $A \equiv \bigsqcap_{1 \leq i \leq n} B_j \in \mathcal{T}$ **do**
  $\mathtt{SUB} \leftarrow \mathtt{SUB} \cup \{\mathtt{REDUCE}_C(\bigsqcap_{1 \leq i \leq n} C_i) | (C_1, ..., C_n) \in \mathtt{SUB}_F(B_1, A) \times ... \times \mathtt{SUB}_F(B_n, A)\}$
**end for**
**for all** $A \equiv \exists r.B \in \mathcal{T}$ **do**
  $\mathtt{SUB} \leftarrow \mathtt{SUB} \cup \{\exists r.C | C \in \mathtt{SUB}_F(B, A), r \in \Sigma\}$
**end for**
**return** $\mathtt{REDUCE}(\mathtt{SUB})$

**Algorithm 2** computing $\mathtt{SUP}_S(A)$ for an $\mathcal{EL}$ TBox $\mathcal{T}$ and a signature $\Sigma$

$\mathtt{SUP} \leftarrow \bigcup \mathtt{SUP}_F(D, A), D \in N_C^+$ such that $\mathcal{T} \models A \sqsubseteq D$
**for all** $A \equiv \bigsqcap_{1 \leq i \leq n} B_j \in \mathcal{T}$ **do**
  $\mathtt{SUP} \leftarrow \mathtt{SUP} \cup \{C | C \in \mathtt{SUP}_F(B_j, A)\}$
**end for**
**for all** $A \equiv \exists r.B \in \mathcal{T}$ **do**
  $\mathtt{SUP} \leftarrow \mathtt{SUP} \cup \{\exists r.\mathtt{REDUCE}_C(\bigsqcap_{C \in \mathtt{SUP}_F(B,A)} C) | r \in \Sigma\}$
**end for**
**return** $\mathtt{REDUCE}(\mathtt{SUP})$

conjuncts. Both, REDUCE and $\mathtt{REDUCE}_C$, can be easily realized using standard reasoning procedures in $\mathcal{EL}_{\mu(\sqcup),\nu}$, which is known to be decidable in *ExpTime* [4]. It is easy to see that, in case of an acyclic TBox $\mathcal{T}$, both algorithms terminate, while, in case of cyclic terminologies, the algorithms do not need to terminate. In Section 7, we show how the termination for general TBoxes can be ensured by introducing fixpoint constructs for concepts involved in terminological cycles with particular properties introduced in the next section.

## 6 Graphs and Trees

To allow for a more intuitive understanding of the cases, in which Algorithms 1 and 2 do not terminate, we introduce the following graphs representing the possible flow of computation of SUB and SUP for a particular TBox $\mathcal{T}$ (independent from a particular signature).

**Definition 4.** *The* SUP*- and* SUB*-graphs* $\mathcal{A}_{\mathtt{SUP}}(\mathcal{T})$ *and* $\mathcal{A}_{\mathtt{SUB}}(\mathcal{T})$ *are defined as*

– $\mathcal{A}_{\mathtt{SUP}}(\mathcal{T}) = (\Gamma_{\mathtt{SUP}}, Q, E_{\mathtt{SUP}})$ *with the set of edge labels* $\Gamma_{\mathtt{SUP}} = N_R \cup \{\sqsubseteq\}$*, the set of states* $Q = N_C$ *and the set of edges* $E_{\mathtt{SUP}} = \{(A, r, B) | A \equiv \exists r.B \in \mathcal{T}\} \cup \{(A, \sqsubseteq, B) | \mathcal{T} \models A \sqsubseteq B\}$*, where* $A, B \in Q$ *and* $r \in \Gamma_{\mathtt{SUP}}$*.*
– $\mathcal{A}_{\mathtt{SUB}}(\mathcal{T}) = (\Gamma_{\mathtt{SUB}}, Q, E_{\mathtt{SUB}})$ *with the set of edge labels* $\Gamma_{\mathtt{SUB}} = N_R \cup \{\sqsupseteq, \sqcap\}$*, the set of states* $Q = N_C$ *and the set of edges* $E_{\mathtt{SUB}} = \{(A, r, B) | A \equiv \exists r.B \in \mathcal{T}\} \cup \{(A, \sqsupseteq, B) | \mathcal{T} \models A \sqsupseteq B\} \cup \{(A, \sqcap, B) | A \equiv B \sqcap C \in \mathcal{T}$ *for any conjunction C of elements from* $Q\}$*, where* $A, B \in Q$ *and* $r \in \Gamma_{\mathtt{SUB}}$*.*

The two graphs can be constructed in linear time after the classification of the normalized TBox is finished. The corresponding subgraphs $\mathcal{A}_{\mathsf{SUP}}(\mathcal{T},\Sigma)$ and $\mathcal{A}_{\mathsf{SUB}}(\mathcal{T},\Sigma)$ representing the computation of SUB and SUP for a particular signature $\Sigma$ can then be obtained from $\mathcal{A}_{\mathsf{SUP}}(\mathcal{T})$ and $\mathcal{A}_{\mathsf{SUB}}(\mathcal{T})$ by omitting all outgoing edges of nodes in $\Sigma$ as well as all edges with labels not from $\Sigma \cup \{\sqsubseteq, \sqsupseteq, \sqcap\}$. Subsequently, concepts in $\Sigma$ form the leaves of the resulting graphs $\mathcal{A}_{\mathsf{SUP}}(\mathcal{T},\Sigma)$ and $\mathcal{A}_{\mathsf{SUB}}(\mathcal{T},\Sigma)$. For $X \in \{\mathsf{SUB}, \mathsf{SUP}\}$, we denote the set of the paths in $\mathcal{A}_X(\mathcal{T},\Sigma)$ from $A$ to $B$ as $L_X(A, B)$ and the set of the intersection-free but possibly cyclic paths as $L^1_X(A, B)$ , i.e., paths not passing any node more than once. As illustrated by the example below, cycles in $\mathcal{A}_{\mathsf{SUP}}(\mathcal{T})$ and $\mathcal{A}_{\mathsf{SUB}}(\mathcal{T})$ do not necessarily coincide. Therefore, both graphs have to be analysed to determine the sets $L^1_{\mathsf{SUP}}(A, B)$ and $L^1_{\mathsf{SUB}}(A, B)$.

*Example 2.* The corresponding SUB- and SUP-graphs of the normalized TBox $\mathcal{T} = \{A_1 \sqsubseteq B, A_1 \equiv A_2 \sqcap A_3, A_3 \sqsubseteq A_2, A \equiv \exists r.B, A_3 \equiv \exists r.B\}$ and the signature $\Sigma = \mathrm{sig}(\mathcal{T})$ are shown in Fig. 1.

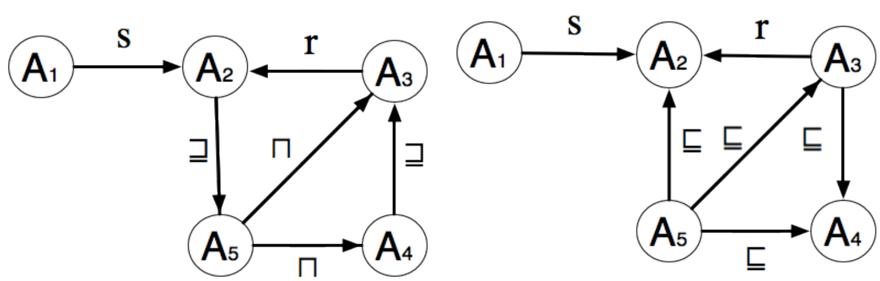


**Fig. 1.** SUB-graph (left) and SUP-graph (right) of $\mathcal{T}$.

Note that, since the nodes from $\Sigma$ are leaves in $\mathcal{A}_{\mathsf{SUP}}(\mathcal{T},\Sigma)$ and $\mathcal{A}_{\mathsf{SUB}}(\mathcal{T},\Sigma)$, $L^1_X(A, B)$ contains only paths formed by nodes of the signature $\overline{\Sigma}$. The presence of concepts with $L^1_X(A, A) \neq \emptyset$ determines whether the computation of SUB and SUP specified for the case of acyclic terminologies terminates. In the following section, we introduce concepts with fixpoint constructs guaranteeing the termination in the presence of such cycles.

## 7 SUB and SUP based on Fixpoint Constructors

In the following, we show how $\mathsf{SUB}(A)$ and $\mathsf{SUP}(A)$ can be computed for cyclic TBoxes based on Algorithms 1 and 2. For this purpose, we now define the values of $\mathsf{SUB}_F(A, B)$ and $\mathsf{SUP}_F(A, B)$ for concepts in cycles, i.e., for any $A \in N_C$ with $L^1_{\mathsf{SUB}}(A, A) \neq \emptyset$ and $L^1_{\mathsf{SUP}}(A, A) \neq \emptyset$, respectively, in such a way that $\mathsf{SUB}(A)$ and $\mathsf{SUP}(A)$ are expressed by the means of a (finite) set of $\mathcal{EL}_{\mu(\sqcup),\nu}$ concepts. In what follows, we denote the two sets of concepts involved in cycles as $\overline{\Sigma}_{C,\mathsf{SUB}} = \{A | L^1_{\mathsf{SUB}}(A, A) \neq \emptyset\}$ and $\overline{\Sigma}_{C,\mathsf{SUP}} = \{A | L^1_{\mathsf{SUP}}(A, A) \neq$

$\emptyset\}$. For each $A_i \in \overline{\Sigma}_{C,\texttt{SUP}}$ with $0 \leq i \leq n$ and each $A_j \in \overline{\Sigma}_{C,\texttt{SUB}}$ with $0 \leq j \leq m$, we introduce two concept variables, one for being used in $\texttt{SUB}(A)$ and one for $\texttt{SUP}(A)$, which we denote with $X(A_i)$ and $Y(A_j)$, respectively. The set of all introduced variables is denoted by $\mathcal{V}_x$ with $x \in \{\texttt{SUP}, \texttt{SUB}\}$. Further, let $C(A_i)$ and $D(A_j)$ be concept expressions possibly containing free variables from $\mathcal{V}_x$, defined as $C(A_i) = \sqcap_{B \in \texttt{SUP}_S(A_i)} B$ and $D(A_j) = \sqcup_{B \in \texttt{SUB}_S(A_j)} B$. Given the values $C(A_i)$ and $D(A_j)$ for each $A_i \in \overline{\Sigma}_{C,\texttt{SUP}}$ and each $A_j \in \overline{\Sigma}_{C,\texttt{SUB}}$, we define

$$N(A_i) = \nu_i X(A_1)...X(A_n).C(A_1),...,C(A_n)$$

$$M(A_j) = \mu_j Y(A_1)...Y(A_m).D(A_1),...,D(A_m).$$

Since free variables are not allowed in the resulting $\texttt{SUB}(A)$ and $\texttt{SUP}(A)$ for any $A \in N_C$, we need to ensure that only the quantified fixpoint expressions, i.e., $M(B)$ or $N(B)$ for any $B \in N_C$, are included into $\texttt{SUB}(A)$ or $\texttt{SUP}(A)$ in Algorithms 1 and 2. For this purpose, we realize two different levels of visibility within $\texttt{SUB}_F(A)$ and $\texttt{SUP}_F(A)$ by the means of the second parameter $B$. This parameter points to the concept, from which $\texttt{SUB}_F$ and $\texttt{SUP}_F$ are called and determines, which of the two visibility levels applies. In case $B$ is involved in the corresponding cycle, e.g., $B \in \overline{\Sigma}_{C,\texttt{SUP}}$ for $A \in \overline{\Sigma}_{C,\texttt{SUP}}$, the internal value of $\texttt{SUB}_F(A)$ and $\texttt{SUP}_F(A)$ is returned, which is given by the corresponding variable $Y(A)/X(A)$ and is only used to compute $C(B)$ or $D(B)$. For $B$ outside the corresponding cycles, we return the complete fixpoint expression in its quantified form, i.e., $M(A)/N(A)$, which is then included into $\texttt{SUB}(A)$ and $\texttt{SUP}(A)$. Therefore, by the means of this additional distinguishing, we ensure that all variables in the resulting $\texttt{SUB}(A)$ and $\texttt{SUP}(A)$ for any $A \in N_C$ are quantified. The full set of distinguishments realized by $\texttt{SUP}_F(A, B)$ and $\texttt{SUB}_F(A, B)$ is given by:

$$\texttt{SUP}_F(A,B) = \begin{cases} A & \text{if } A \in \Sigma \\ X(A) & \text{if } A \in \overline{\Sigma}_{C,\texttt{SUP}}, \\ & B \in \overline{\Sigma}_{C,\texttt{SUP}} \\ N(A) & \text{if } A \in \overline{\Sigma}_{C,\texttt{SUP}}, \\ & B \notin \overline{\Sigma}_{C,\texttt{SUP}} \\ \texttt{SUP}_S(A) & \text{otherwise} \end{cases} \qquad \texttt{SUB}_F(A,B) = \begin{cases} A & \text{if } A \in \Sigma \\ Y(A) & \text{if } A \in \overline{\Sigma}_{C,\texttt{SUB}}, \\ & B \in \overline{\Sigma}_{C,\texttt{SUB}} \\ M(A) & \text{if } A \in \overline{\Sigma}_{C,\texttt{SUB}}, \\ & B \notin \overline{\Sigma}_{C,\texttt{SUB}} \\ \texttt{SUB}_S(A) & \text{otherwise} \end{cases}.$$

Now we summarize the definition of $\texttt{SUP}(A)$ and $\texttt{SUB}(A)$ for the general case of $\texttt{SUB}(A)$ and $\texttt{SUP}(A)$ in $\mathcal{EL}_{\mu(\sqcup),\nu}$.

**Definition 5.** *Let $A \in N_C$. The set of conjuncts for computing $\texttt{MSS}(A)$ and the set of disjuncts for computing $\texttt{MGS}(A)$ in $\mathcal{EL}_{\mu(\sqcup),\nu}$, in symbols $\texttt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ and $\texttt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$, are given by $\texttt{SUP}_F(A, \top)$ and $\texttt{SUB}_F(A, \top)$, respectively, in case $A \in \overline{\Sigma}$, and by $\texttt{SUP}_S(A)$ and $\texttt{SUB}_S(A)$, otherwise.*

We denote the $\mathcal{EL}$ variants of $\texttt{SUP}(A)$ and $\texttt{SUB}(A)$ as $\texttt{SUP}^{\mathcal{EL}}(A)$ and $\texttt{SUB}^{\mathcal{EL}}(A)$. Given a TBox with $\overline{\Sigma}_{C,\texttt{SUB}} \cup \overline{\Sigma}_{C,\texttt{SUB}} = \emptyset$, $\texttt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ and $\texttt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ computed as stated in Definition 5 coincide with $\texttt{SUP}^{\mathcal{EL}}(A)$ and $\texttt{SUB}^{\mathcal{EL}}(A)$.

**Theorem 1 (Termination).** *Let $A \in N_C$. The computation of* $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ *and* $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ *always terminates in at most exponential time.*

*Proof.* We start with $\mathtt{SUP}(A)$ and show that the theorem holds for it. Assume that the input is finite, i.e., $\mathcal{T}$ is finite and contains only finite concept descriptions.

1. Assume that $\overline{\Sigma}_{C,\mathtt{SUP}} = \emptyset$. Algorithm 2 is called for each concept at least once. Since Algorithm 2 itself only contains loops iterating on the input directly, it terminates, if the input is finite and the call of $\mathtt{SUP}_F(A', A)$ for each $A'$ ocurring in the corresponding axioms terminates with a finite result. We can show by induction that Algorithm 2 terminates for an arbitrary concept $A$:
   - If $A$ does not depend on other concepts as stated in Algorithm 2, the result is empty and the algorithm terminates without any processing.
   - If $A$ only depends on concepts $A'$ from $\Sigma$ as stated in Algorithm 2, $\mathtt{SUP}_F(A')$ returns $A'$ itself for each $A'$ and the algorithm terminates.
   - If $A$ only depends on concepts $A'$ from $\Sigma$ or concepts $B'$, for which $\mathtt{SUP}_F(B')$ terminates with a final result.
2. Now assume that $\overline{\Sigma}_{C,\mathtt{SUP}} \neq \emptyset$. $\mathtt{SUP}_F$ encapsulates all concepts in $\overline{\Sigma}_{C,\mathtt{SUP}}$ into a single computational unit with incoming edges from concepts referencing any concept in $\overline{\Sigma}_{C,\mathtt{SUP}}$ and outgoing edges to concepts referenced from any concept in $\overline{\Sigma}_{C,\mathtt{SUP}}$. These two sets of referencing and references concepts are disjoint by definition, i.e., if a concept directly or indirectly references $\overline{\Sigma}_{C,\mathtt{SUP}}$, it is not referenced from $\overline{\Sigma}_{C,\mathtt{SUP}}$. This simplifies the overall computation as follows:
   - On the one hand, we can first compute $N(A)$ for all $A \in \overline{\Sigma}_{C,\mathtt{SUP}}$ and then consider $\mathtt{SUP}_S(A, B)$ for all $B$ referencing $\overline{\Sigma}_{C,\mathtt{SUP}}$ as another case, in which no further computations are required and Algorithm 2 terminates for $B$.
   - On the other hand, we can compute $N(A)$ for all $A \in \overline{\Sigma}_{C,\mathtt{SUP}}$ independently from concepts referencing $\overline{\Sigma}_{C,\mathtt{SUP}}$ by just considering dependencies to concepts in $\overline{\Sigma}_{C,\mathtt{SUP}}$ and concepts not referencing $\overline{\Sigma}_{C,\mathtt{SUP}}$. In this case, either $B \in \overline{\Sigma}_{C,\mathtt{SUP}}$ and the corresponding concept variable is returned, or the computation of $\mathtt{SUP}(B)$ is acyclic and terminates as shown for acylcic terminologies.

Since the structure of $\mathtt{SUB}_F$ and $\mathtt{SUP}_F$ is analogous and $\mathtt{SUB}_S$ also only contains loops iterating on the finite input directly, the argumentation for $\mathtt{SUB}$ is identical. The exponential time is due to the complex conjunction constructs introduced in line 3 of Algorithm 1. □

**Theorem 2 (Correctness $\mathtt{SUP}$ and $\mathtt{SUB}$).** *Let $A \in N_C$. The computed* $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ *and* $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ *satisfy the conditions stated in Definition 2.*

The proof of this theorem is the Section A of the appendix.

**Theorem 3 (UI).** *Let $\mathtt{SUP}(A)$ and $\mathtt{SUB}(A)$ be computed according to Definition 5. Then, $\mathtt{UI}(\mathcal{T}, \Sigma)$ is an $\mathcal{EL}_{\mu(\sqcup),\nu}$ TBox, which always exists and it holds that $\mathtt{UI}(\mathcal{T}, \Sigma) \equiv^{c}_{\Sigma} \mathcal{T}$.*

The proof of this theorem is the Section B of the appendix.

## 8 Existence of UI in $\mathcal{EL}$

Clearly, if $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ and $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ coincide with $\mathtt{SUP}^{\mathcal{EL}}(A)$ and $\mathtt{SUB}^{\mathcal{EL}}(A)$ for all $A \in N_C$, in other words, if $\mathcal{T}$ does not contain pure $\overline{\Sigma}$ cycles, a UI in $\mathcal{EL}$ exists. This would be a sufficient, but not necessary criterion for the existence of a UI. From Definition 3, we can deduce a very general form of criterion requiring the deductive closure of any UI[2] to contain an (arbitrary) finite $\mathcal{EL}$ justification for the set of all non-$\mathcal{EL}$ axioms in the $\mathtt{UI}(\mathcal{T},\Sigma)$. Interestingly, if SUB and SUP are computed using Algorithms 1 and 2, this criterion can be easily checked, since it is equivalent to a very simple criterion, which is an immediate consequence of the following theorem:

**Theorem 4 (Existence).** *Let $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)$ be the subset of $\mathtt{UI}(\mathcal{T},\Sigma)$ containing exactly the $\mathcal{EL}$ axioms of $\mathtt{UI}(\mathcal{T},\Sigma)$ and let $\mathcal{T}'$ be an $\mathcal{EL}$ TBox with $sig(\mathcal{T}') \subseteq \Sigma$ such that $\mathcal{T}' \equiv^c_\Sigma \mathcal{T}$. Then $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma) \equiv \mathcal{T}'$.*

The theorem claims that, if a finite $\mathcal{EL}$ justification for the set of all non-$\mathcal{EL}$ axioms in $\mathtt{UI}(\mathcal{T},\Sigma)$ exists, it is already a subset of it. Subsequently, a UI of $\mathcal{T}$ w.r.t. $\Sigma$ in $\mathcal{EL}$ exists, iff $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma) \models \mathtt{UI}(\mathcal{T},\Sigma)$. The proof of this theorem is based on the ideas stated in Lemmas 2 and 3, which show that there is a close relation between the existence of a UI in $\mathcal{EL}$ and redundancy in $\mathtt{UI}(\mathcal{T},\Sigma)$.

**Lemma 2.** *Let $\mathcal{T}'$ be an $\mathcal{EL}$ TBox with $sig(\mathcal{T}') \subseteq \Sigma$ such that $\mathcal{T}' \equiv^c_\Sigma \mathcal{T}$. Further, let $A \in \overline{\Sigma}_{C,\mathtt{SUP}} \cup \overline{\Sigma}_{C,\mathtt{SUB}}$ with $C_1 \in \mathtt{SUB}(A)$ and $C_2 \in \mathtt{SUP}(A)$. Then there is an $\mathcal{EL}$ concept $C'$ such that*

- $\mathcal{T} \not\models C' \equiv C_1$ *and* $\mathcal{T} \not\models C' \equiv C_2$
- $\mathtt{UI}(\mathcal{T},\Sigma) \models C_1 \sqsubseteq C'$ *and* $\mathtt{UI}(\mathcal{T},\Sigma) \models C' \sqsubseteq C_2$.

*Proof.* Given the form of non-$\mathcal{EL}$ concepts present in $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}$ and $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}$, there is no finite way to express a non-$\mathcal{EL}$ axiom by the means of $\mathcal{EL}$ and $\Sigma$ without introducing new consequences or losing some consequences. To see this, consider the concepts $C_1 = \mu X.(A \sqcup \exists r.X)$ and $C_2 = \nu X.(A \sqcap \exists r.X)$ , which are the simplest possible non-$\mathcal{EL}$ concepts in $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}$ and $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}$, respectively, in case of a normalized TBox $\mathcal{T}$. $C_1$ is semantically equivalent to an infinite disjunction of more and more specific $\mathcal{EL}$ concepts. The language constructs of $\mathcal{EL}$ do not allow us to specify a concept, which captures exactly the subset of the interpretation domain $C_1^{\mathcal{I}}$ in all models. If $C_1 \sqsubseteq C_2$ is a consequence of $\mathcal{T}'$, then there must be an $\mathcal{EL}$ concept $C_1'$, which subsumes $C_1^{\mathcal{I}}$ in all models. Since $\mathcal{T}'$ is in $\mathcal{EL}$ and is finite, it must hold that $\{C_1 \sqsubseteq C_1'\} \models C_1 \sqsubseteq C_1'$,.i.e., the latter inclusion axiom must be derived without any background terminology (e.g., $C_1' = B$ with $\{\exists r.B \sqsubseteq B, A \sqsubseteq B\} \in \mathcal{T}'$). Moreover $C_1' \sqsubseteq C_2$ must have a justification in $\mathcal{T}'$ consisting of finitely many $\mathcal{EL}$ axioms. The same argumentation applies to $C_2$ as a concept with greatest fixpoint constructs, i.e., the deductive closure of a UI contains a finite $\mathcal{EL}$ justification for $C_1 \sqsubseteq C_2$, where $C_1$ can be an arbitrary concept, only if there is a more specific $\mathcal{EL}$ concept $C_2'$ such that $\mathcal{T}' \models C_1 \sqsubseteq C_2'$ has a finite justification in $\mathcal{T}'$ and $C_2' \sqsubseteq C_2$ must be derived without any background terminology. □

[2] The deductive closure is the same for any UI by definition.

The above proof is the first step towards a connection between the redundancy in $\mathtt{UI}(\mathcal{T},\Sigma)$ and the existence of a UI in $\mathcal{EL}$. Since $\{C_1 \sqsubseteq C', C' \sqsubseteq C_2\} \models C_1 \sqsubseteq C_2$ and any minimal justification of $\{C_1 \sqsubseteq C', C' \sqsubseteq C_2\}$ in any $\mathcal{T}'$ does not contain $C_1 \sqsubseteq C_2$, it also holds that $\mathtt{UI}(\mathcal{T},\Sigma) \setminus \{C_1 \sqsubseteq C_2\} \models C_1 \sqsubseteq C_2$. Therefore, if $\mathcal{T}'$ exists, each non-$\mathcal{EL}$ axiom is redundant, i.e., it could be removed from $\mathtt{UI}(\mathcal{T},\Sigma)$ without losing any consequences. In order to prove Theorem 4, we additionally have to show that the dependencies between the axioms in $\mathtt{UI}(\mathcal{T},\Sigma) \setminus \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)$ do not lead to a loss of equivalence between $\mathtt{UI}(\mathcal{T},\Sigma)$ and $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)$. We now consider the possible redundancy in $\mathtt{UI}(\mathcal{T},\Sigma)$. By the means of the functions $\mathtt{REDUCE}$ and $\mathtt{REDUCE}_C$ we have ensured that the sets $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ and $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ do not contain any redundancy. Therefore, it remains to consider the construction of $\mathtt{UI}(\mathcal{T},\Sigma)$ using $\mathtt{SUP}(A)$ and $\mathtt{SUB}(A)$ for $A \in N_C$ as stated in Definition 3. From the definition of $\mathtt{MGS}$ and $\mathtt{MSS}$ follows that the sets $M_1, M_2$ in Definition 3 cannot be redundant if the sets $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ and $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ contain only incomparable elements. Therefore, it remains to consider the redundancy introduced during the construction of $M_3$. We denote by $P_{\overline{\Sigma}} = \{(C_1, C_2) |$ there is $A \in \overline{\Sigma}$ s.t. $C_1 \in \mathtt{SUB}(A), C_2 \in \mathtt{SUP}(A)\}$ the set of all concept pairs relevant for the construction of $M_3$ and the subset of $P_{\overline{\Sigma}}$ containing the "redundant" concept pairs by $\mathcal{R} = \{(C_1, C_2) \in P_{\overline{\Sigma}} | \mathtt{UI}(\mathcal{T},\Sigma) \setminus \{C_1 \sqsubseteq C_2\} \models C_1 \sqsubseteq C_2\}$. I.e., $\mathcal{R}$ is the set of concept pairs that are potentially nonessential for the construction of a UI due to entailment of the corresponding inclusion axiom by the remainder of a UI if the axiom itself is omitted. Due to possible dependencies between the elements of $\mathcal{R}$, there may be several different maximal subsets $M$ of $\mathcal{R}$ such that $\mathtt{UI}(\mathcal{T},\Sigma) \setminus \{C_1 \sqsubseteq C_2 | (C_1, C_2) \in M\} \models \mathtt{UI}(\mathcal{T},\Sigma)$. We denote the set of all such maximal subsets of $\mathcal{R}$ as $\mathcal{R}_{\mathtt{MAX}} = \{M | M \subseteq \mathcal{R}, \mathtt{UI}(\mathcal{T},\Sigma) \setminus \{C_1 \sqsubseteq C_2 | (C_1, C_2) \in M\} \models \mathtt{UI}(\mathcal{T},\Sigma)$, for all $(C_1', C_2') \in P_{\overline{\Sigma}} \setminus M$ holds $\mathtt{UI}(\mathcal{T},\Sigma) \setminus (\{C_1' \sqsubseteq C_2'\} \cup \{C_1 \sqsubseteq C_2 | (C_1, C_2) \in M\}) \not\models \mathtt{UI}(\mathcal{T},\Sigma)\}$. The next lemma states that if a concept pair with at least one non-$\mathcal{EL}$ concept is contained in one set $M \in \mathcal{R}_{\mathtt{MAX}}$, it is contained in all $M \in \mathcal{R}_{\mathtt{MAX}}$.

**Lemma 3.** *Let $A \in \overline{\Sigma}_{C,\mathtt{SUP}} \cup \overline{\Sigma}_{C,\mathtt{SUB}}$ with $C_1 \in \mathtt{SUB}(A)$ and $C_2 \in \mathtt{SUP}(A)$. Further let $M' \in \mathcal{R}_{\mathtt{MAX}}$ such that $(C_1, C_2) \in M'$. Then for each $M \in \mathcal{R}_{\mathtt{MAX}}$ holds $(C_1, C_2) \in M$.*

The proof of this theorem is the Section C of the appendix. Note that all concept pairs with at least one non-$\mathcal{EL}$ concept are contained in the intersection of $\mathcal{R}_{\mathtt{MAX}}$, iff $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma) \equiv \mathcal{T}'$. As a consequence of the above two lemmas and the fact that for any $(C_1, C_2) \in \mathcal{R}$ there exists at least one $M \in \mathcal{R}_{\mathtt{MAX}}$, it is sufficient to check whether all concept pairs with at least one non-$\mathcal{EL}$ concept are contained in $\mathcal{R}$ to determine whether the $\mathcal{T}'$ in Theorem 4 exists.

## 9 Summary

In this paper, we provided *ExpTime* algorithms for computing uniform interpolants of general $\mathcal{EL}$ terminologies preserving all $\mathcal{EL}$ concept inclusions for a particular signature based on the notion of *most general subconcepts* and *most specific superconcepts*. We showed that such interpolants can always be expressed in logic $\mathcal{EL}_{\mu(\sqcup),\nu}$—an extension of $\mathcal{EL}$ with least fixpoint and greatest fixpoint constructors $\mu, \nu$ as well as the disjunction used only on the left-hand side of concept inclusions. We also stated the exact existence criteria for an $\mathcal{EL}$ interpolant and showed how it can be obtained from the corresponding interpolant expressed in $\mathcal{EL}_{\mu(\sqcup),\nu}$.

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## A Proof of Theorem 2

The algorithms presented in this paper are based on Proof Theory. The used Gentzen-stile proof system shown in Fig. 2 has been derived analogously to the proof system for Horn-$\mathcal{SHIQ}$ terminologies presented in [7]. In principle, the proof system by Kazakov can also be used in the subsequent proofs, however it requires a different normalization (e.g., encoding all $\exists r.A \sqsubseteq B$ as $A \sqsubseteq \forall r^{-}.B$), which we prefer not to use for simplicity reasons. Instead, we derive rules fitting our normal form. The proof system is sound and complete for classification in logic $\mathcal{ELH}^{r}_{\text{ran}}$, which is a Horn-extension of $\mathcal{EL}$ by role inclusions and the range operator ran. For a role $r$, $\text{ran}(r)$ can be used in concept inclusion axioms in addition to the already introduced $\mathcal{EL}$ constructs. the reason for the proof system being complete for $\mathcal{ELH}^{r}_{\text{ran}}$ is the planned extension of the presented forgetting approach to $\mathcal{ELH}^{r}_{\text{ran}}$ in future work. Since the proof system is complete for classification, an arbitrary subsumption between two non-atomic concepts is entailed by the TBox $\mathcal{T}$, if it is derivable in the presented proof system after the corresponding definition for the non-atomic concept on the left- and the right-hand side of the subsumption has been added to the TBox. It is easy to see that adding a definition for a concept description $C$ with $\text{sig}(C) \in \text{sig}(\mathcal{T})$ by introducing a fresh concept symbol yields a conservative extension of $\mathcal{T}$. In the following, we denote the resulting TBox after inserting a definition for a concept $C$ or a set $M$ of concepts into $\mathcal{T}$ and applying the normalization to it as $\texttt{EXT}(\mathcal{T}, C)$ and $\texttt{EXT}(\mathcal{T}, M)$, respectively. $\texttt{SYM}(C)$ denotes the corresponding fresh concept symbol introduced to define $C$ and $\texttt{DEF}(C)$ denotes the concept $D$ such that $\texttt{SYM}(C) \equiv D \in \texttt{EXT}(\mathcal{T}, C)$. Note that $\texttt{DEF}(C)$ is not necessarily syntactically equivalent to $C$ due to normalization. We further denote the set of all fresh concept symbols introduced by the latter extension of $\mathcal{T}$ as $N_D = \text{sig}(\texttt{EXT}(\mathcal{T}, C))/\text{sig}(\mathcal{T})$. Moreover, $\bowtie$ denotes one of $\{\sqsubseteq, \equiv\}$. Until further notice, we use $N_C$ and $N_C^+$ to refer to the signature of $\mathcal{T}$ (not the signature of $\texttt{EXT}(\mathcal{T}, C)$).

**Lemma 4 (Soundness and Completeness).** *Let $\mathcal{T}$ be a normalized $\mathcal{ELH}^{r}_{ran}$ TBox, $A, B \in N_C$. Then $\mathcal{T} \models A \sqsubseteq B$, iff $\mathcal{T} \vdash A \sqsubseteq B$.*

*Proof.* While the soundness of the proof system (if-direction) is readily checked for each rule, the proof of completeness is more sophisticated. In order to show the only-if-direction of the lemma, we assume for any $A, B$ that $\mathcal{T} \vdash A \sqsubseteq B$ does not hold and construct a model of $\mathcal{T}$, in which there is an individual $a \in A^{\mathcal{I}}/B^{\mathcal{I}}$. The model is constructed analogously to [1]:

- $\Delta^{\mathcal{I}} := \{A \sqcap \text{ran}(r) | A \in N_C^+, r \in N_R\} \cup \{*\}$
- $A^{\mathcal{I}} := \{B \sqcap \text{ran}(r) \in \Delta^{\mathcal{I}} | \mathcal{T} \vdash B \sqcap \text{ran}(r) \sqsubseteq A, r \in N_R\} \cup \{*\}$
- $r^{\mathcal{I}} := \{(A \sqcap \text{ran}(r_j), B \sqcap \text{ran}(r_i)) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | \mathcal{T} \vdash A \sqsubseteq \exists r.B, r_i \sqsubseteq r \in \mathcal{T}, r_j \in N_R\} \cup \{(*, A \sqcap \text{ran}(r_i)) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | r_i \sqsubseteq r \in \mathcal{T}\}$

The given interpretation is a model of $\mathcal{T}$, since it satisfies all its axioms:

- $B_1 \sqsubseteq B_2 \in \mathcal{T}$ are satisfied, since for each $A \sqcap \text{ran}(r) \in B_1^{\mathcal{I}}$ holds $\mathcal{T} \vdash A \sqcap \text{ran}(r) \sqsubseteq B_1$ and, therefore $A \in B_2^{\mathcal{I}}$ due to the rule GCI. To see this, consider the antecedent $\mathcal{T} \vdash A \sqcap \text{ran}(r) \sqsubseteq B_1, B_2 \sqsubseteq B_2$ and the consequence $\mathcal{T} \vdash A \sqcap \text{ran}(r) \sqsubseteq B_2$.

$$\frac{}{C \sqsubseteq C}(\textsc{Ax}) \quad \frac{}{C \sqsubseteq \top}(\textsc{AxTop})$$

$$\frac{D \sqsubseteq E}{C \sqcap D \sqsubseteq E}(\textsc{AndL})$$

$$\frac{C \sqsubseteq E \quad C \sqsubseteq D}{C \sqsubseteq D \sqcap E}(\textsc{AndR})$$

$$\frac{C \sqsubseteq D_1 \quad D_2 \sqsubseteq D}{C \sqsubseteq D}(\text{GCI}) \quad \text{where } D_1 \bowtie D_2 \in \mathcal{T}$$

$$\frac{C \sqsubseteq \exists r_1.D \quad \text{ran}(r_2) \sqsubseteq D_1}{C \sqsubseteq \exists r_1.(D \sqcap D_1)}(\textsc{Ran}) \quad \text{where } r_1 \sqsubseteq r_2 \in \mathcal{T}$$

$$\frac{C \sqsubseteq \exists r_1.D_1 \quad D_1 \sqcap \text{ran}(r_1) \sqsubseteq D_2 \quad \exists r_2.D_2 \sqsubseteq D}{C \sqsubseteq D}(\textsc{Dom})$$

$$\frac{\text{ran}(r_2) \sqsubseteq D}{\text{ran}(r_1) \sqsubseteq D}(\textsc{RanSub}) \quad \text{where } r_1 \sqsubseteq r_2 \in \mathcal{T}$$

**Fig. 2.** Gentzen-style proof system for normalized $\mathcal{ELH}^r_{\text{ran}}$ terminologies in the presence of GCIs.

- $A_1 \equiv B1 \sqcap ... \sqcap B_n \in \mathcal{T}$ are satisfied, since for each direction, $A_1 \sqsubseteq B1 \sqcap ... \sqcap B_n$ and $B1 \sqcap ... \sqcap B_n \sqsubseteq A_1$ holds the same condition as above. For each $A \sqcap \text{ran}(r) \in A_1^{\mathcal{I}}$ holds $A \sqcap \text{ran}(r) \in (B1 \sqcap ... \sqcap B_n)^{\mathcal{I}}$ again due to the rule GCI. For the direction $A \sqcap \text{ran}(r) \in (B1 \sqcap ... \sqcap B_n)^{\mathcal{I}}$ implies $A \sqcap \text{ran}(r) \in A_1^{\mathcal{I}}$, we first obtain $A \sqcap \text{ran}(r) \in B_1^{\mathcal{I}} \cap ... \cap B_n^{\mathcal{I}}$, which implies $\mathcal{T} \vdash A \sqcap \text{ran}(r) \sqsubseteq B_1, ..., A \sqcap \text{ran}(r) \sqsubseteq B_n$. By rule AndR, we obtain $\mathcal{T} \vdash A \sqcap \text{ran}(r) \sqsubseteq B1 \sqcap ... \sqcap B_n$, and by GCI follows $\mathcal{T} \vdash A \sqcap \text{ran}(r) \sqsubseteq A_1$. By definition of $\mathcal{I}$, $A \sqcap \text{ran}(r) \in A_1^{\mathcal{I}}$.
- $B_1 \equiv \exists r.B_2 \in \mathcal{T}$ is shown for each direction as follows. First, recall that $(\exists r.B_2)^{\mathcal{I}} = \{a|(a,b) \in r^{\mathcal{I}}, b \in B_2^{\mathcal{I}}\}$. The argumentation for the direction $B_1^{\mathcal{I}} \subseteq (\exists r.B_2)^{\mathcal{I}}$ is as above. Using the rule GCI we obtain $\mathcal{T} \vdash A \sqcap \text{ran}(r) \sqsubseteq \exists r.B_2$, from which we conclude that $(A \sqcap \text{ran}(r), B_2 \sqcap \text{ran}(r)) \in r^{\mathcal{I}}$, and since $B_2 \sqcap \text{ran}(r) \in B_2^{\mathcal{I}}$ by definition, we obtain $A \sqcap \text{ran}(r) \in (\exists r.B_2)^{\mathcal{I}}$. For the opposite direction, we first conclude from $C \in (\exists r.B_2)^{\mathcal{I}}$ that there is $B \sqcap \text{ran}(r) \in B_2^{\mathcal{I}}$ such that $(C, B \sqcap \text{ran}(r)) \in r^{\mathcal{I}}$. These conclusions imply that $\mathcal{T} \vdash C \sqsubseteq \exists r.B$ and $\mathcal{T} \vdash B \sqcap \text{ran}(r) \sqsubseteq B_2$. Given that $B_1 \equiv \exists r.B_2 \in \mathcal{T}$, we can first obtain $\mathcal{T} \vdash \exists r.B_2 \sqsubseteq B_1$ and then employ the rule Dom to obtain $\mathcal{T} \vdash C \sqsubseteq B_1$. From the definition of $\mathcal{I}$ follows $C \in B_1^{\mathcal{I}}$.
- $r_1 \sqsubseteq r_2 \in \mathcal{T}$ follows from the definition of $r^{\mathcal{I}}$. If there are $(C, B \sqcap \text{ran}(r_1)) \in r_1^{\mathcal{I}}$, then $\mathcal{T} \vdash C \sqsubseteq \exists r_1.B$. After the application of Dom, we obtain $\mathcal{T} \vdash C \sqsubseteq \exists r_2.B$, and, by the definition of $\mathcal{I}$, $(C, B \sqcap \text{ran}(r_1)) \in r_2^{\mathcal{I}}$.
- $A \equiv \text{ran}(r) \in \mathcal{T}$ is shown analogously to $B_1 \equiv \exists r.B_2 \in \mathcal{T}$ for each direction. Assume that $A_1 \sqcap \text{ran}(r_1) \in A^{\mathcal{I}}$, i.e., $\mathcal{T} \vdash A_1 \sqcap \text{ran}(r_1) \sqsubseteq A$. By rule GCI we again obtain $\mathcal{T} \vdash A_1 \sqcap \text{ran}(r_1) \sqsubseteq \text{ran}(r)$, and, by the definition of $r^{\mathcal{I}}$, $(*, A_1 \sqcap \text{ran}(r_1)) \in r^{\mathcal{I}}$. Recall that $\text{ran}(r)^{\mathcal{I}} = \{b|(a,b) \in r^{\mathcal{I}}\}$. Therefore, also $A_1 \sqcap \text{ran}(r_1) \in \text{ran}(r)^{\mathcal{I}}$. For the opposite direction, note that using AndL and *RanSub*, we can obtain from $\mathcal{T} \vdash \text{ran}(r) \sqsubseteq A$ the consequence $A' \sqcap \text{ran}(r_i) \sqsubseteq A$ with an arbitrary $A'$ and $r_i \sqsubseteq r$.

Since $\mathrm{ran}(r)^{\mathcal{I}}$ only contains elements of the form $A' \sqcap \mathrm{ran}(r_i)$ with $r_i \sqsubseteq r$, and, by definition, $A^{\mathcal{I}}$ contains all elements $A' \sqcap \mathrm{ran}(r_i)$ such that $\mathcal{T} \vdash A' \sqcap \mathrm{ran}(r) \sqsubseteq A$ and $r_i \sqsubseteq r$, it is easy to see that all elements of $\mathrm{ran}(r)^{\mathcal{I}}$ are subsumed by $A^{\mathcal{I}}$. □

**Lemma 5 (Cut Elimination).** *Let $\mathcal{T}$ be a normalized $\mathcal{ELH}^r_{ran}$ TBox, $C_1, C_2$ and $C_3$ $\mathcal{ELH}^r_{ran}$ concepts. If $\mathcal{T} \vdash C_1 \sqsubseteq C_2$ and $\mathcal{T} \vdash C_2 \sqsubseteq C_3$, then also $\mathcal{T} \vdash C_1 \sqsubseteq C_3$.*

The proof is standard, and is structurally equivalent to that in [6].

For the subsequent theorems concerning the properties of SUB, we introduce the following auxiliary function $\mathtt{Pre} : N_C : 2^{2^{N_C}}$, which allows us for any atomic concept $A$ to refer to all its subconcepts of a particular form, namely a conjunction of a minimal required set of $N_C$ concepts. For each such conjunction, the set of its conjuncts is an element of Pre.

**Definition 6.** *Let $\mathcal{T}$ be an $\mathcal{ELH}^r_{ran}$ TBox and $A \in N_C$. $\mathtt{Pre}(A)$ is the smallest set with the following properties:*

- $\{A\} \in \mathtt{Pre}(A)$.
- *For each $M \in \mathtt{Pre}(A)$ and each $B \in M$, if there is $B \equiv B_1 \sqcap ... \sqcap B_n \in \mathcal{T}$, then also $(M/\{B\}) \cup \{B_1, ..., B_n\} \in \mathtt{Pre}(A)$.*
- *For each $M \in \mathtt{Pre}(A)$ and each $B \in M$, if there is $\mathcal{T} \models B' \sqsubseteq B$, then also $(M/\{B\}) \cup \{B'\} \in \mathtt{Pre}(A)$.*

Note that the sets $M \in \mathtt{Pre}(A)/\{\{A\}\}$ do not contain $A$, since no equivalent $N_C$ concepts are present in normalized terminologies. Therefore the dependencies between $N_C$ concepts corresponding to Pre are acyclic. The concepts constructed by conjunction from the sets $M \in \mathtt{Pre}(A)/\{\{A\}\}$ can be elements of the set constructed in line 3 of Algorithm 1. However, since the construction depends on $\Sigma$, and there are also concept definitions of the form $\exists r.B$, they are not the only possible elements of this set.

In the following, we will make use of the limited interaction between concepts in $N_C$ and concepts in $N_D$ stated in the next lemma.

**Lemma 6.** *Let $\mathcal{T}$ be an $\mathcal{ELH}^r_{ran}$ TBox and $C$ an $\mathcal{ELH}^r_{ran}$ concept with $sig(C) \subseteq sig(\mathcal{T})$, but not syntactically present in $\mathcal{T}$. Let $B' \in N_D$. Then*

1. *Each sequent of the form $C \sqsubseteq B'$, which has $B' \sqsubseteq B'$ as a direct antecedent, is either a result of* AndL *such that $C = C' \sqcap B'$, or a result of* GCI *w.r.t. $B' \equiv \mathtt{DEF}(B') \in \mathtt{EXT}(\mathcal{T}, C)$.*
2. *Each sequent of the form $B' \sqsubseteq C$, which has $B' \sqsubseteq B'$ as a direct antecedent, is either a result of* AndR *such that $C = C' \sqcap B'$, or a result of* GCI *w.r.t. $B' \equiv \mathtt{DEF}(B') \in \mathtt{EXT}(\mathcal{T}, C)$.*

*Proof.* Note that concepts in $N_D$ are only allowed one definition and occur only within definitions of other concepts from $N_D$. Therefore, the interaction between concepts in $N_C$ and concepts in $N_D$ is limited to the interaction between their definitions and $N_C$. The only rules that admit the above form of antecedent and consequence are in case of $C \sqsubseteq B'$ AndL, which requires the consequence to have a conjunction on the left-hand side, and GCI, in which case the only axiom that could be used for the rule application is $B' \equiv \texttt{DEF}(B') \in \texttt{EXT}(\mathcal{T}, C)$. In case of $B' \sqsubseteq C$, the situation is equivalent, except that AndR instead of AndL is the applied rule. □

In the following, we consider only proofs with a minimal proof tree, i.e., proof tree not containing any sequent twice on the same path. A direct consequence of the above lemma and the fact that concepts in $N_D$ are only allowed one definition and occur only within definitions of other concepts from $N_D$ is the following theorem applying to finite minimal proofs.

**Theorem 5.** *Let $\mathcal{T}$ be an $\mathcal{ELH}^r_{ran}$ TBox and $C$ an $\mathcal{ELH}^r_{ran}$ concept with $sig(C) \subseteq sig(\mathcal{T})$, but not syntactically present in $\mathcal{T}$. Let $A \in N_D$. Then the following sequents are not derivable without an application of the rule* GCI *w.r.t. $A \equiv C' \in \texttt{EXT}(\mathcal{T}, C)$:*

1. $A \sqsubseteq \exists r.D$,
2. $A \sqsubseteq A'$,
3. $\exists r.D \sqsubseteq A$,
4. $A' \sqsubseteq A$,
5. $A \sqsubseteq \bigsqcap_{1 \leq i \leq n} A'_i$ ,
6. $A'' \sqsubseteq A$,

*where $A' \in N_C, A'_i \in N_C \cup N_D, r \in N_R, D$ and $\mathcal{ELH}^r_{ran}$ concept, and $A'' \in N_D$ such that the rule* GCI *w.r.t. $A'' \equiv C'' \in \texttt{EXT}(\mathcal{T}, C)$ is not part of the proof for $A'' \sqsubseteq A$.*

*Proof.* We show the proof for the first statement by induction on the proof length. Assume that $\texttt{EXT}(\mathcal{T}, C) \vdash A \sqsubseteq \exists r.D$. Then rules Dom, Ran and GCI could be the last applied rules. If Dom was the last applied rule, then $A \sqsubseteq \exists r'.D'$ for some $r'$ and $D'$ has also a proof. If for the derivation of $A \sqsubseteq \exists r'.D'$ an application of GCI w.r.t. $A \equiv C' \in \texttt{EXT}(\mathcal{T}, C)$ was required, then also for the derivation of $A \sqsubseteq \exists r.D$. If Ran was the last applied rule, then the theorem also follows from the induction hypothesis, since $A \sqsubseteq \exists r'.D'$ for some $r'$ and $D'$ is again a sequent before the application of the rule. If GCI w.r.t. $B \equiv C' \in \texttt{EXT}(\mathcal{T}, C)$ for some concept $B$ was the last applied rule, the sequent before the application was one of the following:

- $A \sqsubseteq B$. Ax, GCI and Dom could be the last applied rules. If Dom was the last applied rule, then $A \sqsubseteq \exists r'.D'$ for some $r'$ and $D'$ has also a proof and the theorem follows from the induction hypothesis. If Ax was the last applied rule, then $B = A$ and therefore, the application of GCI w.r.t. $A \equiv C' \in \texttt{EXT}(\mathcal{T}, C)$ took place. If GCI was the last applied rule, then the situation is the same as above and, since the proof is finite, one of the discussed cases will occur.
- $A' \sqsubseteq \texttt{DEF}(B)$. If $\texttt{DEF}(B) = \exists r'.D'$ for some $r' \in N_R$ and $D' \in N_C \cup N_D$, the theorem follows again from the induction hypothesis. If $\texttt{DEF}(B) = \bigsqcap_{1 \leq i \leq n} B'_i$ for $B'_i \in N_C \cup N_D$, then, additionally AndR is applicable. The sequents before its application had the form $A \sqsubseteq B'_i$. If $B'_i = A$, then Ax is applicable. However there must be also other concepts in $\bigsqcap_{1 \leq i \leq n} B'_i$, since concepts from $N_D$ do not occur in non-equivalence axioms and equivalence axioms with one atomic concept on both sides do not occur in normalized terminologies. For this reason, there will be also at least one concept $B'_i \neq A$ and the sequent $A \sqsubseteq B'_i$ must also have a proof. The other rules applicable are GCI and Dom, however the situation is the same as discussed and, therefore, the theorem follows from the induction hypothesis.

The correctness of the remaining statement can be shown analogously.

**Theorem 6.** *Let $\mathcal{T}$ be an $\mathcal{ELH}^r_{ran}$ TBox and $C$ an $\mathcal{ELH}^r_{ran}$ concept with $sig(C) \subseteq sig(\mathcal{T})$ such that*

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.D_k$$

*Further let $A \in N_D$ and assume that $\mathtt{DEF}(C) = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} A'_k$ with $A'_k \in N_D, A'_k \equiv \exists r_k.D'_k \in \mathtt{EXT}(\mathcal{T}, C)$, where $D'_k$ is either in $N_C$ or in $N_D$. Further, assume $\mathtt{EXT}(\mathcal{T}, C) \vdash \mathtt{SYM}(C) \sqsubseteq A$. Then one of the following is true:*

1. *There is $A'_k = A$.*
2. *Rule* GCI *w.r.t. $A \equiv C' \in \mathcal{T}$ is part of the proof.*

*Proof.* We show this theorem by induction on the length of the proof. Only GCI, AndL, Dom could be the last applied rules within a proof of the sequent $\mathtt{SYM}(C) \sqsubseteq A$. If Dom was the last applied rule, then $\exists r.D \sqsubseteq A$ for some $r$ and $D$ also has a proof. Due to lemma 6, GCI w.r.t. $A \equiv C' \in \mathtt{EXT}(\mathcal{T}, C)$ is part of the proof. If AndL was the last applied rule and the sequent before the application has the same form,i.e., the left-hand side is a conjunction, then the theorem follows from the induction hypothesis. Otherwise, if $A'$ is the only remaining conjunct and it is in $N_C$, by Lemma 6, GCI w.r.t. $A \equiv C' \in \mathtt{EXT}(\mathcal{T}, C)$ is part of the proof. If $A'$ is not in $N_C$, only GCI, Ax, Dom could be the last applied rules. If Dom was the last applied rule, then the situation is as above and condition 2 is true. Ax requires that $A' = A$, which corresponds to condition 1. In the case of GCI w.r.t. $B \equiv C' \in \mathtt{EXT}(\mathcal{T}, C)$ for some concept $B$, the sequent before the application was one of the following:

- $A' \sqsubseteq B, \mathtt{DEF}(B) \sqsubseteq A$. If $\mathtt{DEF}(B) = \exists r.D$ for some $R \in N_R$ and $D \in N_C \cup N_D$, then by Lemma 6, GCI w.r.t. $A \equiv C' \in \mathtt{EXT}(\mathcal{T}, C)$ is part of the proof for the sequent $\exists r.D \sqsubseteq A$. If $\mathtt{DEF}(B) = \bigsqcap_{1 \leq i \leq n} B'_i$ for $B'_i \in N_C \cup N_D$, i follows from the induction hypothesis, that either condition 2 is true, in which case also the theorem is true for $\bigsqcap_{1 \leq i \leq m+n} A'_i \sqsubseteq A$, or there is $B'_i = A$. In the latter case, $B'_i$ corresponds to one of $A'_k$. This is due to the fact that for $A' \sqsubseteq B$ also applies the induction hypothesis and either there is $A'_k = B$, in which case the only definition for $B$ has the form $\exists r_k.D'_k$ which contradicts with the previous assumption about $B$, or the condition 2 holds for $A' \sqsubseteq B$, in which case the proof is not minimal or not final.
- $A' \sqsubseteq \mathtt{DEF}(B), B \sqsubseteq A$. The theorem follows from the last statement of Lemma 5 (that GCI w.r.t. at least one of the definitions $\mathtt{DEF}(B), \mathtt{DEF}(A)$ is required) and the assumption that the proof is minimal and final.

If GCI was applied w.r.t. $B \equiv C' \in \mathtt{EXT}(\mathcal{T}, C)$ for some concept $B$, then the situation is the same as above, when $A'$ is the only remaining conjunct and GCI was the last applied rule. □

**Theorem 7.** *Let $\mathcal{T}$ be a normalized $\mathcal{ELH}^r_{ran}$ TBox, $r_k \in N_R$, $A$ and $A_j \in N_C$, $C$ an $\mathcal{ELH}^r_{ran}$ concept, $D_k$ for $1 \leq k \leq m$ a set of $\mathcal{ELH}^r_{ran}$ concepts. Assume that*

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.D_k$$

*and $\mathcal{T} \models C \sqsubseteq A$. Then at least one of the following conditions is true:*

*(A1)* *There is $A_j$ such that $\mathcal{T} \models A_j \sqsubseteq A$*
*(A2)* *There is a subset $M$ of $\{A_j | 1 \leq j \leq n\}$ such that $A \equiv \bigsqcap_{A_j \in M} A_j \in \mathcal{T}$.*
*(A3)* *There are $r_k, D_k$ and there exist $r' \in N_R, B' \in N_C$ such that $\mathcal{T} \models r_k \sqsubseteq r'$, $\mathcal{T} \models D_k \sqcap ran(r_k) \sqsubseteq B'$ and $A \equiv \exists r'.B' \in \mathcal{T}$.*
*(A4)* *There is a set of $N_C$ concepts $M \in \mathtt{Pre}(A)/\{\{A\}\}$ such that for each $B' \in M$ holds $\bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.D_k \sqsubseteq B'$ and at least one of the conditions [A1]-[A3] holds w.r.t. $B'$ and the latter inclusion axiom.*

*Proof.* If $\mathcal{T} \models C \sqsubseteq A$, then $\mathtt{EXT}(\mathcal{T}, C) \vdash \mathtt{DEF}(C) \sqsubseteq A$. We consider all rules, that could have been the last rule applied in order to obtain the above sequent and show by induction on the length of the proof that, in each case, at least one of [A1]-[A4] is true. Rules AxTop, AndR, Ran do not allow for a concept from $N_C$ on the right-hand side. In case of the rules Ax and RanSub, the condition [A1] is an immediate consequence. It remains to consider the rules AndL, Dom and GCI. If $\mathtt{DEF}(C)$ is a conjunction $\bigsqcap_{1 \leq j \leq n+m} B_j$, AndL could be the last applied rule. If one of the theorem conditions is true for the antecedent, it is also true for the consequence, since $\mathcal{T} \models \bigsqcap_{1 \leq j \leq n+m} B_j \sqsubseteq \bigsqcap_{1 \leq j \leq n+m-1} B_j$ and all $B_j$ from the smaller conjunction are in $\mathtt{DEF}(C)$ as well.

If GCI w.r.t. $C'_1 \equiv C'_2 \in \mathtt{EXT}(\mathcal{T}, C)$ for some concept $C'_1, C'_2$ was the last applied rule, then one of $C'_1, C'_2$ has to be atomic in normalized TBoxes. Assume that $C'_1$ is atomic. If $C'_1$ is in $N_D$ and the proof is assumed to be final and minimal, then by Theorem 5, it can only be one of $B_j$. In this case, $C'_2 = \exists r_k.\mathtt{SYM}(D_k)$, and if condition [A3] or [A4] holds for $\exists r_k.\mathtt{SYM}(D_k) \sqsubseteq A$, it also holds for $C \sqsubseteq A$. Note that $\exists r_k.\mathtt{SYM}(D_k) \sqsubseteq A$ is a special case of the theorem, in which a subset of rules applicable to obtain the sequent $\mathtt{DEF}(C) \sqsubseteq A$ could be the last applied rule. Therefore, it follows from the induction hypothesis, that one of the condition [A3] or [A4] holds for $\exists r_k.\mathtt{SYM}(D_k) \sqsubseteq A$.

If $C'_1 \in N_C$, then for $\mathtt{DEF}(C) \sqsubseteq A$ holds [A4], if for $\mathtt{DEF}(C) \sqsubseteq C'_1$ holds one of the four conditions. If $C'_2$ is atomic but not $C'_1$, it cannot be in $N_D$ by Theorem 5. One of the following was the sequent before the application of the rule:

- $\mathtt{DEF}(C) \sqsubseteq \exists r.D, C'_2 \sqsubseteq A$ for some $r \in N_R$ and $D \in N_C \cup N_D$, then, in addition to the rules AndL, Dom and GCI, Ran could be the last applied rule. In this case, if one of the theorem conditions [A3] or [A4] w.r.t. $C'_2$ holds for $\mathtt{DEF}(C) \sqsubseteq \exists r.D'$ with $D' = D \sqcap \mathrm{ran}(r)$, then they also hold for $\mathtt{DEF}(C) \sqsubseteq \exists r.D$ w.r.t. $C'_2$, in which case [A4] holds for $\mathtt{DEF}(C) \sqsubseteq A$, since $\mathtt{Pre}(C'_2) \subseteq \mathtt{Pre}(A)$.
- $\mathtt{DEF}(C) \sqsubseteq \bigsqcap_{1 \leq i \leq n} B'_i, C'_2 \sqsubseteq A$ for $B'_i \in N_C \cup N_D$. In this case, AndR could be the last applied rule, in addition to three rules applicable for the original sequent. If AndR was the last applied rule, then [A4] holds for $\mathtt{DEF}(C) \sqsubseteq A$, if for each sequent $\mathtt{DEF}(C) \sqsubseteq B'_i$ holds one of [A1]- [A4], since $\mathtt{Pre}(B'_1) \times ... \times \mathtt{Pre}(B'_n) \subseteq \mathtt{Pre}(A)$.

If Dom was the last applied rule, then the sequents before the rule application were $\mathtt{DEF}(C) \sqsubseteq \exists r_1.D_1, \mathrm{ran}(r_1) \sqcap D_1 \sqsubseteq D_2, \exists r_2.D_2 \sqsubseteq A$. For the last sequent, [A3] or [A4] hold for $\exists r_2.D_2 \sqsubseteq A$ by the induction hypothesis. Since $r_1 \sqsubseteq r_2 \in \mathcal{T}$ and $\mathtt{EXT}(\mathcal{T}, C) \vdash \mathrm{ran}(r_1) \sqcap D_1 \sqsubseteq D_2$, the same condition holds also for $\exists r_1.D_1 \sqsubseteq A$. If $\mathtt{DEF}(C)$ consists only of a single $A_k$ and $\mathtt{EXT}(\mathcal{T}, C) \vdash \mathtt{DEF}(C) \sqsubseteq \exists r_1.D_1$ was derived using only GCI w.r.t. $\mathtt{DEF}(A_k)$ and Ran, Dom, then the same condition holds for $\mathtt{DEF}(C) \sqsubseteq A$, since in this case $r_k \sqsubseteq r_1 \in \mathcal{T}$ and $\mathcal{T} \models \mathrm{ran}(r_k) \sqcap D_k \sqsubseteq D_1$. Otherwise, GCI was applied w.r.t. $\mathtt{DEF}(A')$ for some $A' \in N_C$ for the same reasons as already discussed in case on GCI. In this case, [A4] holds for $\mathtt{DEF}(C) \sqsubseteq A$, since from the induction hypothesis follows that one of [A1]- [A4] holds for $\mathtt{DEF}(C) \sqsubseteq A'$ and $\mathtt{Pre}(A') \subseteq \mathtt{Pre}(A)$. □

Before we can prove the correctness of computing MGS, we introduce the following structure, which is used as a basis for the induction in the subsequent proof. In the following, we denote the set of sequents of a proof $p$ as $S(p)$ and the subset of $S(p)$ containing only sequents of the form $C \sqsubseteq A$ for $A \in N_C$ and an $\mathcal{ELH}^r_{\text{ran}}$ concept $C$ as $S'(p)$. We refer to the right-hand side concept of a sequent $b$ as $\mathtt{Right}(b)$.

**Definition 7.** *Let $\mathcal{T}$ be an $\mathcal{ELH}^r_{ran}$ TBox, $A \in N_C$ and $C$ an $\mathcal{ELH}^r_{ran}$ concept. Further let $p$ be a proof for a sequent $a = C \sqsubseteq A$. The corresponding $\mathtt{RPG}(p)$ is a tuple $(N, E)$ with the set of nodes $N \subseteq S'(p)$ and the set of edges $E \subseteq S'(p) \times S'(p)$. We further distinguish the elements of $E$ into elements of $E_{A3}$ and elements of $E_{A4}$, which are disjoint subsets of $E$. $N$ and $E$ are minimal sets with the following properties:*

- *$a \in N$.*
- *For each $b \in N$ and each $c \in S'(p)$ such that for $b$ holds condition [A3] from Theorem 7 w.r.t. $c$ holds: $c \in N$ and $(b, c) \in E_{A3}$.*
- *For each $b \in N$ and each $M \in \mathtt{Pre}(\mathtt{Right}(b))$ such that for $b$ holds condition [A4] from Theorem 7 w.r.t. $M$ holds: $c \in N$ and $(b, c) \in E_{A4,M}$ for each $c \in M$.*

While the $\mathtt{RPG}(p)$ contains all possible dependencies, we are interested in tree-shaped subgraphs of $\mathtt{RPG}(p)$ not containing any proper subsets of edge sets $E_{A4,M}$, but always containing at least one outgoing edge, if a sequent does not fulfill [A1] or [A2].

**Definition 8.** *Let $\mathcal{T}$ be an $\mathcal{ELH}^r_{ran}$ TBox, $A \in N_C$ and $C$ an $\mathcal{ELH}^r_{ran}$ concept. Further let $p$ be a proof for a sequent $a = C \sqsubseteq A$ and $(N, E)$ the corresponding $\mathtt{RPG}(p)$. A tree with the set of nodes $N_T \subseteq N$ and the set of edges $E_T \subseteq E$ is a RP-tree (Reverse Proof tree), if $N_T$ and $E_T$ are minimal sets such that $a \in N_T$ and for each $b \in N_T$ either one of [A1] and [A2] holds and $\{(b, b') | (b, b') \in E_T\} = \emptyset$, or exactly one of the following holds:*

- *There is $b' \in N_T$ such that $(b, b') \in E_{T,A3}$.*
- *There is $M \in \mathtt{Pre}(\mathtt{Right}(b))$ such that $b' \in N_T$ and $(b, b') \in E_{T,A4,M}$ for each $b' \in M$.*

Now we state that for each proof for a sequent of the form $C \sqsubseteq A$ for $A \in N_C$ and an $\mathcal{ELH}^r_{\text{ran}}$ concept $C$ there exists such a tree.

**Lemma 7.** *Let $\mathcal{T}$ be an $\mathcal{ELH}^r_{ran}$ TBox, $A \in N_C$ and $C$ an $\mathcal{ELH}^r_{ran}$ concept. Further let $p$ be a proof for a sequent $a = C \sqsubseteq A$ and $(N, E)$ the corresponding $\mathtt{RPG}(p)$. Then there exists a finite RP-tree such that for each $(a, b) \in E_T$ holds that $b$ occurs in $p$ before $a$.*

We can now prove the first part of Theorem 2.

**Theorem 8.** *Let $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ be computed as stated in Definition 5. Then the following holds:*

- *For each $C' \in \mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ holds $sig(C') \subseteq \Sigma$;*
- *For each $A \in N_C$ and each $\mathcal{EL}$ concept $C$ with $\mathcal{T} \models C \sqsubseteq A$ and $sig(C) \subseteq \Sigma$ there is a concept $C'$ such that $\mathcal{T} \models C \sqsubseteq C'$ and $\mathtt{MGS}(A) = C' \sqcup C''$ for some concept $C''$ .*

*Proof.* To prove this theorem, we consider the general concept $C = \bigsqcap_{1\leq j\leq n} A_j \sqcap \bigsqcap_{1\leq k\leq m} \exists r_k.D_k$ with $A_j \in N_C, r_k \in N_R$ and assume that $\mathrm{sig}(C) \subseteq \Sigma$ and $\mathcal{T} \models C \sqsubseteq A$. We consider the elements of $\mathtt{SUB}_S(B)$, which, in case $B \in \overline{\Sigma}_{C,\mathtt{SUB}}$, are used to compose the quantified fixpoint concept $M(B)$, and, otherwise, coincide with the elements of $\mathtt{SUB}(B)$. If $\mathtt{MGS}$ contains fixpoint concepts, then it can be represented by an infinite disjunction of $\mathcal{EL}$ concepts which has the same semantics. We show the correctness of the above theorem by iterating through the cases discussed in Theorem 7. We use induction on the depth of the corresponding RP-tree. According to the definition of RP-trees, each node $a$ is a leaf, has one outgoing edge from $E_{T,A3}$, or has the complete set of outgoing edges $E_{T,A4,M}$ for a particular $M \in \mathtt{Pre}(A\mathtt{Right}(a))$. First, we show that, for sequents fulfilling one of the conditions [A1] and [A2], i.e., the leaves of the RP-tree, the theorem is true.

(A1) There is $A_j$ such that $\mathcal{T} \models A_j \sqsubseteq A$. In this case, $A_j$ is included into $\mathtt{SUB}_S(A)$ in line 1, and therefore $C' = A$ with $C \sqsubseteq C'$.

(A2) There is a subset $M$ of $\{A_j | 1 \leq i \leq n\}$ such that $A \equiv \bigsqcap_{A_i\in M} A_i \in \mathcal{T}$. In this case, $\bigsqcap_{A_i\in M} A_i$ is included into $\mathtt{SUB}_S(A)$ in line 3, which implies that $C' = \bigsqcap_{A_i\in M} A_i$ and $C \sqsubseteq C'$.

Now we assume that $a = C \sqsubseteq A$ is a sequent within the RP-tree and the theorem is true for all its successor sequents. Assume that $a$ has an outgoing $E_{T,A3}$-edge to $b = D_k \sqcap \mathrm{ran}(r_k) \sqsubseteq D''$, i.e., there are $r_k, D_k$ and there exist $r', D'$ such that $\mathcal{T} \models r_k \sqsubseteq r'$, $\mathcal{T} \models D_k \sqcap \mathrm{ran}(r_k) \sqsubseteq D'$ and $A \equiv \exists r'.D' \in \mathcal{T}$. Then, the theorem is also true for $a$ for the following reasons. If $A \equiv \exists r'.D' \in \mathcal{T}$ and $\exists r'.D'$ is a $\Sigma$ concept, it is included into $\mathtt{SUB}_S(A)$ in line 6, which implies that $C' = \exists r'.D'$ and $C \sqsubseteq C'$. If $\exists r'.D'$ is not a $\Sigma$ concept, either $r' \in \overline{\Sigma}$ or $D'$ is not a $\Sigma$ concept or both. If $r' \in \overline{\Sigma}$, there is $s$ with $\mathcal{T} \models r_k \sqsubseteq s \sqsubseteq r'$ such that, according to line 6, $\{\exists s.D'' | D'' \in \mathtt{SUB}_F(D')\} \subseteq \mathtt{SUB}_S(A)$. If, as assumed above, there is a $D'' \in \mathtt{SUB}(D')$ such that $\mathcal{T} \models D_k \sqcap \mathrm{ran}(r_k) \sqsubseteq D''$, it holds that $\mathcal{T} \models \exists r_k.D_k \sqsubseteq \exists s.D''$, since $\mathcal{T} \models r_k \sqsubseteq s$. Subsequently, $C' = \exists s.D''$.

Assume that $a$ has the complete set of outgoing edges $E_{T,A4,M}$ for a particular $M \in \mathtt{Pre}(A)$ and for each successor the theorem holds. Each element of $\mathtt{Pre}(A)$ is either a $N_C$ concept $B$ such that $\mathcal{T} \models B \sqsubseteq A$, or a conjunction of $N_C$ concepts which is a direct definition of $A$ or obtained from such a definition by replacing concepts by their $N_C$ definitions or $N_C$ subconcepts. For $N_C$ concepts $B$, $\mathtt{SUB}_F(B)$ , and therefore the corresponding $C''$ with $\mathcal{T} \models C \sqsubseteq C''$ is directly included into $\mathtt{SUB}_S(A)$ and corresponds to $C'$. For $N_C$ concepts occurring within the $N_C$ definition of $A$, the conjunction of such concepts $C''_i$ is included in line 3 into $\mathtt{SUB}_S(A)$, and therefore, $C' = \bigsqcap_{1\leq i\leq n} C''_i$. Since the elements $M_i$ of $\mathtt{Pre}(A)$ form a tree w.r.t. $\sqsubseteq$-relation applied to the corresponding concepts constructed from each $M_i$, we can show by induction that, if, for some $M_i \in \mathtt{Pre}(A)$ holds that there is a concept $C''_i$ such that $\mathcal{T} \models C \sqsubseteq C''_j$ for each $B_j \in M_i$, then there is also such a concept $C'$ with $\mathcal{T} \models C \sqsubseteq C'$, which is an element of the disjunction $\mathtt{MGS}(A)$ due to line 1 or line 3. □

**Theorem 9.** *Let $\mathcal{T}$ be a normalized $\mathcal{ELH}^r_{ran}$ TBox, $r_k \in N_R, A \in N_C$ and $D_k, C$ $\mathcal{ELH}^r_{ran}$ concepts. Assume that $C = \exists r_k.D_k$ and $\mathcal{T} \models A \sqsubseteq C$. Then there is $B \in N_C$ such that $\mathcal{T} \models A \sqsubseteq B$ and $B \equiv \exists r'.D' \in \mathcal{T}$ for some $r' \in N_R$ and $D'$ an $\mathcal{ELH}^r_{ran}$ concept with $\mathcal{T} \models r' \sqsubseteq r_k$ and $\mathcal{T} \models ran(r') \sqcap D' \sqsubseteq D_k$.*

*Proof.* We will make use of the limited interaction between concepts in $N_C$ and concepts in $N_D$ to show that $B \in N_C$. From the latter fact also follows that $B \equiv \exists r'.D' \in \mathcal{T}$, since the above extension of $\mathcal{T}$ does not allow new definitions for concepts in $N_C$. In the following, we use induction on the length of the proof. In principle, the last applied rule for deriving the sequent $A \sqsubseteq \exists r_k.D_k$ could be any rule except AndR, AndL and Ax, AxTop. First, we consider the rules Dom, RanSub, and Ran, that could be applied last to obtain $\mathtt{EXT}(\mathcal{T}, C) \vdash A \sqsubseteq \exists r_k.D_k$. Assume that the condition of the theorem holds for the antecedent. In case of RanSub, $A = \mathrm{ran}(r_2)$. If we assume that the condition was true for $\mathrm{ran}(r_1) \sqsubseteq \exists r_k.D_k$, then it is also true for $\mathrm{ran}(r_2) \sqsubseteq \exists r_k.D_k$, since $\mathcal{T} \models \mathrm{ran}(r_2) \sqsubseteq B$. And in case of Ran, if we assume, that the condition holds for $A \sqsubseteq \exists r_k.D$, then it also holds for $A \sqsubseteq \exists r_k.D_k$, since $\mathcal{T} \models \mathrm{ran}(r') \sqcap D' \sqsubseteq D$ implies $\mathcal{T} \models \mathrm{ran}(r') \sqcap D' \sqsubseteq \mathrm{ran}(r_2) \sqcap D$ due to $\mathcal{T} \models \mathrm{ran}(r') \sqsubseteq \mathrm{ran}(r_2)$. If Dom was the last applied rule, then the condition holds, if either $\mathcal{T} \models r_2 \sqsubseteq r_k$ and $\mathcal{T} \models \mathrm{ran}(r_2) \sqcap D_2 \sqsubseteq D_k$ and the condition holds for the sequent $A \sqsubseteq \exists r_1.D_1$, or for the sequent $\exists r_2.D_2 \sqsubseteq \exists r_k.D_k$ holds the corresponding condition, i.e., there is $B \in N_C$ such that $\mathcal{T} \models \exists r_2.D_2 \sqsubseteq B$ and $B \equiv \exists r'.D'$ for some $r' \in N_R$ and $D'$ an $\mathcal{ELH}^r_{\mathrm{ran}}$ concept with $\mathcal{T} \models r' \sqsubseteq r_1$ and $\mathcal{T} \models \mathrm{ran}(r') \sqcap D' \sqsubseteq D_k$. The same situation appears also when the rule GCI is applied, therefore we now consider GCI and the corresponding situation.

If GCI was the last applied rule, then the sequent before the application of the rule was $\mathcal{T} \vdash A \sqsubseteq D_1 \quad D_2 \sqsubseteq \exists r_k.D_k$. Note that in normalized TBoxes, at least one of $D_1, D_2$ is an atomic concept. If $D_2$ is atomic, then $D_2 \in N_C$ due to Theorem 5 and the assumed minimality of proofs, and the condition of the theorem holds, if it holds for $D_2 \sqsubseteq \exists r_k.D_k$. If $D_2$ is atomic (and therefore also in $N_C$) and $D_2$ is of the form $\exists r.D$, then either $\mathcal{T} \models r \sqsubseteq r_1$ and $\mathcal{T} \models \mathrm{ran}(r) \sqcap D \sqsubseteq D_k$, in which case $D_1$ is such a concept $B$ as required in the condition of the theorem, or the same condition holds for $\mathcal{T} \vdash \exists r.D \sqsubseteq \exists r_k.D_k$, i.e., there is $B \in N_C$ such that $\mathcal{T} \models \exists r.D \sqsubseteq B$ and $B \equiv \exists r'.D'$ for some $r' \in N_R$ and $D'$ an $\mathcal{ELH}^r_{\mathrm{ran}}$ concept with $\mathcal{T} \models r' \sqsubseteq r_1$ and $\mathcal{T} \models \mathrm{ran}(r') \sqcap D' \sqsubseteq D_k$. The last rule applied to derive the sequent $\mathcal{T} \vdash \exists r.D \sqsubseteq \exists r_k.D_k$ could be one of the rules GCI, Ran, Dom, which can be considered in the same way as for the sequent $A \sqsubseteq \exists r_k.D_k$. Additionally, it could be Ax, in which case $r_k = r$ and $D_k = D$, and $D_1$ again corresponds to the concept $B$ specified in the condition of the theorem. If $D_2$ is a conjunction of atomic concepts $\bigsqcap_{1 \le i \le n} A_i$, then the conditions holds, if there is $B \in N_C$ such that $\mathcal{T} \models \bigsqcap_{1 \le i \le n} A_i \sqsubseteq B$ and $B \equiv \exists r'.D'$ for some $r' \in N_R$ and $D'$ an $\mathcal{ELH}^r_{\mathrm{ran}}$ concept with $\mathcal{T} \models r' \sqsubseteq r_1$ and $\mathcal{T} \models \mathrm{ran}(r') \sqcap D' \sqsubseteq D_k$. In this case, also the same rules could have been applied last except for RanSub, and additionally AndL. If AndL was the last applied rule, and we assume that the condition holds for the antecedent $\bigsqcap_{1 \le i \le n-1} A_i \sqsubseteq \exists r_k.D_k$, then the condition also holds for $\bigsqcap_{1 \le i \le n-1} A_i \sqcap A_n \sqsubseteq \exists r_k.D_k$, since $\mathcal{T} \models \bigsqcap_{1 \le i \le n-1} A_i \sqcap A_n \sqsubseteq \bigsqcap_{1 \le i \le n-1} A_i$. □

We can now prove the second part of Theorem 2.

**Theorem 10.** *Let $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ be computed as stated in Definition 5. Then the following holds:*

- *For each $C' \in \mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ holds $sig(C') \subseteq \Sigma$;*
- *For each $A \in N_C$ and each $\mathcal{EL}$ concept $C$ with $\mathcal{T} \models A \sqsubseteq C$ and $sig(C) \subseteq \Sigma$ there is a concept $C'$ such that $\mathcal{T} \models C' \sqsubseteq C$ and $\mathtt{MSS}(A) = C' \sqcap C''$ for some concept $C''$ .*

*Proof.* To prove this theorem, consider the general concept $C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.D_k$. Then, $\mathcal{T} \models A \sqsubseteq C$, iff for each conjunct $C_i$ of $C$ holds $\mathcal{T} \models A \sqsubseteq C_i$. We consider the elements of $\mathtt{SUP}_S(B)$, which, in case $B \in \overline{\Sigma}_{C,\mathtt{SUP}}$, are used to compose the quantified fixpoint concept $N(B)$, and, otherwise, coincide with the elements of $\mathtt{SUP}(B)$. If MSS contains fixpoint concepts, then it can be represented by an infinite conjunction of $\mathcal{EL}$ concepts which has the same semantics as MSS. If $C_i$ is an element of $N_C$, then according to line 1, it is included into $\mathtt{SUP}_S(A)$ and, therefore, corresponds to $C'$. If $C_i = \exists r_k.D_k$, then, by Theorem 9 , there is $B \in N_C$ such that $\mathcal{T} \models A \sqsubseteq B$ and $B \equiv \exists r'.D'$ for some $r' \in N_R$ and $D'$ an $\mathcal{ELH}^r_{\text{ran}}$ concept with $\mathcal{T} \models r' \sqsubseteq r_k$ and $\mathcal{T} \models \text{ran}(r') \sqcap D' \sqsubseteq D_k$. If $B \in \Sigma$, then $B \in \mathtt{SUP}_S(A)$ and $C' = B$. Otherwise, according to line 1, $\mathtt{SUP}_S(A)$ contains the elements of $\mathtt{SUP}_S(B)$. In case of $\overline{\Sigma}_{C,\mathtt{SUP}}$ concepts, the corresponding variable $X(B)$ is a direct element of $\mathtt{SUP}_S(A)$, which implies that all direct elements of $\mathtt{SUP}_S(B)$ will be elements of the infinite conjunction represented by $\mathtt{SUP}(A)$. If $\exists r'.D'$ is a $\Sigma$ concept, then $C' = \exists r'.D'$. Otherwise, either $r' \in \overline{\Sigma}$ or $D'$ is not a $\Sigma$ concept or both. If $r' \in \overline{\Sigma}$, there is $s$ with $\mathcal{T} \models r' \sqsubseteq s \sqsubseteq r_k$ such that, according to line 6, $\exists s.(\bigsqcap_{C'' \in \mathcal{C}} C'') \in \mathtt{SUP}_S(A)$ for $\mathcal{C} = \{C'' \in \mathtt{SUP}_F(B') | \mathcal{T} \models \text{ran}(r') \sqcap D' \sqsubseteq B'\}$. Since $\mathcal{T} \models \text{ran}(r') \sqcap D' \sqsubseteq D_k$ and $D_k \in \Sigma$, $D_k$ will be an element of $\mathcal{C}$. Therefore, $C' = \exists s.(\bigsqcap_{C'' \in \mathcal{C}} C'')$. □

## B Proof of Theorem 3

**Lemma 8.** *Let $C, D$ two $\mathcal{EL}$ concepts and $r \in N_R$ and assume that $C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.D_k$ with $r_k \in N_R, A_j \in N_C$ and $D_k$ a set of $\mathcal{EL}$ concepts. $\mathcal{T} \models C \sqsubseteq \exists r.D$, if one of the following conditions is true:*

1. *There are $r_k, D_k$ such that $\mathcal{T} \models r_k \sqsubseteq r$ and $\mathcal{T} \models ran(r_k) \sqcap D_k \sqsubseteq D$.*
2. *There is $B \in N_C$ such that $\mathcal{T} \models C \sqsubseteq B$ and $\mathcal{T} \models B \sqsubseteq \exists r.D$.*

*Proof.* We use the extension $\mathtt{EXT}(\mathcal{T}, \{C, \exists r.D\})$ and proof the theorem using the proof system presented above by induction on the length of the proof. By Theorem 4, $\mathtt{EXT}(\mathcal{T}, \{C, \exists r.D\}) \vdash \mathtt{SYM}(C) \sqsubseteq \mathtt{SYM}(\exists r.\mathtt{SYM}(D))$. In the following, we denote $\mathtt{EXT}(\mathcal{T}, \{C, \exists r.D\})$ as $\mathcal{T}'$, $\mathtt{SYM}(C)$ simply with $C_s$, $\mathtt{SYM}(\exists r.\mathtt{SYM}(D))$ with $D_s$ and $\mathtt{SYM}(D)$ with $D'$.

The last rules applied to derive the sequent $C_s \sqsubseteq D_s$ could be Dom, GCI and Ax. In case of Ax, condition 1 is an immediate consequence. In case of GCI w.r.t. $C_1 \bowtie C_2$ for some concepts $C_1, C_2$, at least one of the concepts has to be atomic. If $C_1$ is atomic and $C_1 \in N_D$, then from Theorem 6 follows that it has to be one of $A_k \equiv \mathtt{DEF}(\exists r_k.D_k)$. In this case, the theorem is true, since $C_2 = \mathtt{DEF}(\exists r_k.D_k)$ and condition 1 or 2 for $C_2 \sqsubseteq D_s$ follows from the induction hypothesis. If condition 1 holds for $C_2 \sqsubseteq D_s$ , then it also holds for $C_s \sqsubseteq D_s$, since $\mathcal{T} \models r_k \sqsubseteq r$ and $\mathcal{T} \models \text{ran}(r) \sqcap D_k \sqsubseteq D$. If 2 holds for $C_2 \sqsubseteq D_s$, i.e., there is $B$ with the corresponding properties, then it also holds that $\mathcal{T} \models C_s \sqsubseteq B$ and therefore, condition 2 holds. If $C_1 \in N_C$, $C_1$ corresponds to $B$ and the the theorem is true.

If $C_2$ is atomic, then from Theorem 6 follows that it can not be in $N_D$, if we assume that the proof is finite and minimal. If $C_2 \in N_C$, then the theorem is also true, since $\mathcal{T} \models B \sqsubseteq D_s$.

If Dom was the last applied rule, then the two sequents $C_s \sqsubseteq \exists r_1.D_1, \exists r_2.D_2 \sqsubseteq D_s$ have been derived before the rule application. From the induction hypothesis follows that for each of the two sequents, one of the theorem conditions holds. If for both holds 1, then also for $C_s \sqsubseteq D_s$, since $r_1 \sqsubseteq r_2 \in \mathcal{T}$ and $\text{ran}(r_1) \sqcap D_1 \sqsubseteq D_2$ was the third sequent derived before the application of Dom. If at least for one of the two sequents condition 2 holds, then it also holds for $C_s \sqsubseteq D_s$, since $C_s \sqsubseteq B \sqsubseteq D_s$. □

We now restate and prove the claim of Theorem 3.

**Theorem 11.** *Let* $\mathtt{UI}(\mathcal{T}, \Sigma)$ *be constructed as in Definition 3 using* $\mathtt{SUP}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$ *and* $\mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(A)$*. Then* $\mathtt{UI}(\mathcal{T}, \Sigma) \equiv^c_\Sigma \mathcal{T}$*.*

*Proof.* By definition, $\mathtt{UI}(\mathcal{T}, \Sigma) \equiv^c_\Sigma \mathcal{T}$, if for all $\mathcal{EL}$ concepts $C, D$ with $\text{sig}(C) \cup \text{sig}(D) \subseteq \Sigma$ holds $\mathcal{T} \models C \sqsubseteq D$, iff $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D$. We start with the if-direction and consider two general concepts $C, D$ with $\text{sig}(C) \cup \text{sig}(D) \subseteq \Sigma$. Assume that $C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.D_k$ with $r_k \in N_R, A_j \in N_C$ for $1 \leq j \leq n$ and $D_k$ a set of $\mathcal{EL}$ concepts. If $D$ is a conjunction, then $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D$ holds, iff for each conjunct $D_i$ of $D$ holds $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D_i$.

If $D_i \in N_C$, then, by Definition 3, $C_i \sqsubseteq D_i \in \mathtt{UI}(\mathcal{T}, \Sigma)$ for all $C_i \in \mathtt{SUB}^{\mathcal{EL}_{\mu(\sqcup),\nu}}(D_i)$. Therefore $\mathtt{UI}(\mathcal{T}, \Sigma) \models \mathtt{MGS}(D_i) \sqsubseteq D_i$. By Theorem 8 there is a concept $C'$ such that $\mathcal{T} \models C \sqsubseteq C'$ and $\mathtt{MGS}(D_i) = C' \sqcup C''$ for some concept $C''$. Therefore, also $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D_i$.

If $D_i \notin N_C$, but has the form $\exists r'.D'$ for some $r' \in N_R$ and an $\mathcal{EL}$ concept $D'$ with $\text{sig}(D') \subseteq \Sigma$ , then we can show by induction on the role depth of $C$ that $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D_i$. If the role depth of $C$ is 0, i.e., it does not contain any existential quantifiers, then condition 2 of Lemma 8 holds, i.e., there is $B \in N_C$ such that $\mathcal{T} \models C \sqsubseteq B \sqsubseteq \exists r'.D'$. If $B \in \overline{\Sigma}$, then by Definition 3, $C' \sqsubseteq D' \in \mathtt{UI}(\mathcal{T}, \Sigma)$ for any $C' \in \mathtt{SUB}(B)$ and $D' \in \mathtt{SUP}(B)$. By Theorems 8 and 10, there are such $C'$ and $D'$ with $\mathcal{T} \models C \sqsubseteq C'$ and $\mathcal{T} \models D' \sqsubseteq \exists r'.D'$. If $\mathcal{T} \models C \sqsubseteq C'$ implies $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq C'$ and $\mathcal{T} \models D' \sqsubseteq \exists r'.D'$ implies $\mathtt{UI}(\mathcal{T}, \Sigma) \models D' \sqsubseteq \exists r'.D'$, then also $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq \exists r'.D'$. If $B \in \Sigma$, then by Definition 3, $C' \sqsubseteq B \in \mathtt{UI}(\mathcal{T}, \Sigma)$ for any $C' \in \mathtt{SUB}(B)$ and $B \sqsubseteq D' \in \mathtt{UI}(\mathcal{T}, \Sigma)$ for any $D' \in \mathtt{SUP}(B)$. Therefore, also in this case, $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq \exists r'.D'$ if $\mathcal{T} \models C \sqsubseteq C'$ implies $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq C'$ and $\mathcal{T} \models D' \sqsubseteq \exists r'.D'$ implies $\mathtt{UI}(\mathcal{T}, \Sigma) \models D' \sqsubseteq \exists r'.D'$.

To see that $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq C'$ for $C' \in \mathtt{SUB}(B)$, we refer to Theorem 7, by which either A1 or A2 holds for $C \sqsubseteq B$, in which case $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq C'$ is a direct consequence of the inclusion axioms formed using $\mathtt{SUB}(B)$, or there is $M \in \mathtt{Pre}(B)$ such that A1 or A2 holds for each $B_i \in M$. In the second case, we can easily determine the corresponding $M' \in \mathtt{Pre}(B)$ such that $M' \subseteq \Sigma$ and $\bigsqcap_{A \in M'} A \in \mathtt{SUB}(B)$ by replacing all elements of $M$ by their subconcepts or conjunctions until all elements are in $\Sigma$. Since this corresponds to the procedure of Algorithm 1, we can set $\bigsqcap_{A \in M'} A = C'$, in which case the statement $\mathtt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq C'$ is a direct consequence.

To see that $\mathtt{UI}(\mathcal{T}, \Sigma) \models D' \sqsubseteq \exists r'.D'$ for $D' \in \mathtt{SUP}(B)$, we again refer to Lemma 8, in which again only condition 1 is possible. Since the signature of $\mathcal{T}$ is finite, there is a finite set of atomic concepts $A_i$ with $1 \leq i \leq n$ ordered by $\sqsubseteq$ such that $\mathcal{T} \models A_i \sqsubseteq A_{i+1} \sqsubseteq \exists r'.D'$ for each $i$ and there are $r'', D''$ such that $\exists r''.D'' \in \mathtt{SUP}(A_n)$ and for $\exists r'.D' \sqsubseteq \exists r'.D'$ holds A3 of Theorem 7. Then $\mathtt{UI}(\mathcal{T}, \Sigma) \models B \sqsubseteq A_n$, from which follows $\mathtt{UI}(\mathcal{T}, \Sigma) \models B \sqsubseteq \exists r'.D'$ and $\mathtt{UI}(\mathcal{T}, \Sigma) \models \exists r'.D' \sqsubseteq \exists r'.D'$ is easy to see, since the role hierarchy of $\mathtt{UI}(\mathcal{T}, \Sigma)$ contains all role subsumptions for roles in $\Sigma$ and it holds that $\mathtt{UI}(\mathcal{T}, \Sigma) \models r'' \sqsubseteq r'$.

Now assume that the role depth of $C$ is not 0. Then, by Lemma 8, also condition 1 is possible, in which case there are $r_k, D_k$ such that $\mathcal{T} \models r_k \sqsubseteq r$ and $\mathcal{T} \models \text{ran}(r_k) \sqcap D_k \sqsubseteq D'$. By induction hypothesis, $\texttt{UI}(\mathcal{T}, \Sigma) \models \text{ran}(r_k) \sqcap D_k \sqsubseteq D'$. Then also $\texttt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D_i$, since the role hierarchy contains all role subsumptions for roles in $\Sigma$ and it holds that $\texttt{UI}(\mathcal{T}, \Sigma) \models r_k \sqsubseteq r'$. It remains to consider the case that the role depth of $C$ is not 0 and the condition 2 of Lemma 8 holds. Here, we show by induction on the depth of the corresponding RP-tree that $\texttt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D_i$, the leaves of which are nodes $A_k$ with A1 or A2 of Theorem 7 applying to $C \sqsubseteq A_k$. Assume that $D_i$ is such a leaf. Then one of the following is the case:

(A1) There is $A_j$ such that $\mathcal{T} \models A_j \sqsubseteq D_i$, in which case also $\texttt{UI}(\mathcal{T}, \Sigma) \models A_j \sqsubseteq D_i$, since $A_j \sqsubseteq D_i \in \texttt{UI}(\mathcal{T}, \Sigma)$. Therefore, $\texttt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D_i$.
(A2) There is a subset $M$ of $\{A_j | 1 \leq j \leq n\}$ such that $D_i \equiv \bigsqcap_{A_j \in M} A_j \in \mathcal{T}$. In this case, $\bigsqcap_{A_j \in M} A_j \sqsubseteq D_i \in \texttt{UI}(\mathcal{T}, \Sigma)$ and, therefore, $\texttt{UI}(\mathcal{T}, \Sigma) \models C \sqsubseteq D_i$.

Now we assume that a=$C' \sqsubseteq D_i$ is a sequent within the RP-tree and the $\texttt{UI}(\mathcal{T}, \Sigma) \models b$ is true for all its successor sequents $b = C'' \sqsubseteq C'$. Assume that $a$ has an outgoing $E_{T,A3}$ edge to $b = D_k \sqcap \text{ran}(r_k) \sqsubseteq D''$, i.e., there are $r_k, D_k$ and there exist $r', D'$ such that $\mathcal{T} \models r_k \sqsubseteq r'$, $\mathcal{T} \models D_k \sqcap \text{ran}(r_k) \sqsubseteq D'$ and $A \equiv \exists r'.D' \in \mathcal{T}$. Then also $\texttt{UI}(\mathcal{T}, \Sigma) \models a$, since the role hierarchy contains all role subsumptions for roles in $\Sigma$, therefore it holds that $\texttt{UI}(\mathcal{T}, \Sigma) \models r_k \sqsubseteq r'$.

Assume that $a$ has the complete set of outgoing edges $E_{T,A4,M}$ for a particular $M \in \texttt{Pre}(D_i)$ and for each successor $b_k$ holds $\texttt{UI}(\mathcal{T}, \Sigma) \models b_k$. Each element of $\texttt{Pre}(D_i)$ is either a $N_C$ concept $B$ such that $\mathcal{T} \models B \sqsubseteq D_i$, or a conjunction of $N_C$ concepts which is a direct definition of $D_i$ or obtained from such a definition by replacing concepts by their $N_C$ definitions or $N_C$ subconcepts.

For $N_C$ concepts $B$, $\texttt{SUB}_F(B)$ , and therefore the corresponding $C''$ with $\mathcal{T} \models C \sqsubseteq C''$ is directly included into $\texttt{SUB}_S(A)$ and corresponds to $C'$. For $N_C$ concepts occurring within the $N_C$ definition of $A$, the conjunction of such concepts $C''_i$ is included in line 3 into $\texttt{SUB}_S(A)$, and therefore, $C' = \bigsqcap_{1 \leq i \leq n} C''_i$. Since the elements $M_i$ of $\texttt{Pre}(A)$ form a tree w.r.t. $\sqsubseteq$ relation applied to the corresponding concepts constructed from each $M_i$, we can show by induction that, if, for some $M_i \in \texttt{Pre}(A)$ holds that there is a concept $C''_i$ such that $\mathcal{T} \models C \sqsubseteq C''_j$ for each $B_j \in M_i$, then there is also such a concept $C'$ with $\mathcal{T} \models C \sqsubseteq C'$, which is an element of the disjunction $\texttt{MGS}(A)$ due to line 1 or line 3.

For the only-if direction, is easy to see, that all inclusion axioms contained in $\texttt{UI}(\mathcal{T}, \Sigma)$ are consequences of $\mathcal{T}$. □

## C Proof of Lemma 3

**Lemma 9.** *Let $A \in \overline{\Sigma}_{C,\texttt{SUP}} \cup \overline{\Sigma}_{C,\texttt{SUB}}$, $C_1 \in \texttt{SUB}(A)$ and $C_2 \in \texttt{SUP}(A)$. Let $\mathcal{T}'$ be an $\mathcal{EL}$ TBox with sig($\mathcal{T}'$) $\subseteq \Sigma$ such that $\mathcal{T}' \equiv^c_\Sigma \mathcal{T}$. Then there is a concept $A' \in N_C \setminus \{A\}$ and there are two $\mathcal{EL}$ concepts $C'_1 \in \texttt{SUB}(A')$ and $C'_2 \in \texttt{SUP}(A')$ such that $\mathcal{T} \models C_1 \sqsubseteq C'_1$ and $\mathcal{T} \models C'_2 \sqsubseteq C_2$.*

*Proof.* If $\mathcal{T}' \models C_1 \sqsubseteq C_2$, then a consequence-driven classification of the TBox $\texttt{EXT}(\mathcal{T}', \{C_1, C_2\}$ would yield $\texttt{SYM}(C_1) \sqsubseteq \texttt{SYM}(C_2)$. Since our proof system in Fig. 2 does not allow for fixpoint constructs, we extend it with the following rules:

$$\frac{C \sqsubseteq D_1}{C \sqsubseteq D_1 \sqcup D_2}(\textsc{OrR})$$

$$\frac{}{C(\nu X.C(X)) \sqsubseteq \nu X.C(X)}(\textsc{Gfp}1)$$

$$\frac{}{\mu X.C(X) \sqsubseteq C(\mu X.C(X))}(\textsc{Lfp}1)$$

$$\frac{}{\nu X.C(X) \sqsubseteq C(\nu X.C(X))}(\textsc{Gfp}2)$$

$$\frac{}{C(\mu X.C(X)) \sqsubseteq \mu X.C(X)}(\textsc{Lfp}2)$$

$$\frac{C(A) \sqsubseteq A}{\mu X.C(X) \sqsubseteq A}(\textsc{Lfp}\text{A}1)$$

$$\frac{A \sqsubseteq C(A)}{A \sqsubseteq \nu X.C(X)}(\textsc{Gfp}\text{A}1)$$

$$\frac{\mu X.C(X) \sqsubseteq A}{\mu X.C(X) \sqsubseteq C(A)}(\textsc{Lfp}\text{A}2)$$

$$\frac{A \sqsubseteq \nu X.C(X)}{C(A) \sqsubseteq \nu X.C(X)}(\textsc{Gfp}\text{A}2)$$

$$\frac{C_1(A) \sqsubseteq A \quad C_2(A) \sqsubseteq A}{\mu X.(C_1(X) \sqcup C_2(X)) \sqsubseteq A}(\textsc{OrLfp})$$

In [11] it has been shown that each concept using the mutual fixpoint constructor has a corresponding non-mutual representation, therefore, we restrict the proof system to the more simple rules for non-mutual fixpoint constructors. In the above rules, $C(A)$ denotes a concept, in which $A$ occurs at least once within an existential quantification. The correctness of the rules can be seen easily, the completeness can be shown analogously to Theorem 4 by constructing the canonical model and showing that, if for some $A, B \in N_C$ the subsumption cannot be deduced, then the subset relationship also does not hold in the model. First, note that, on the one hand, $\mathcal{T}'$ is an $\mathcal{EL}$ TBox. Therefore, the inclusion $C_1 \sqsubseteq C_2$ is derived using only $\mathcal{EL}$ axioms. On the other hand, the two concepts $C_1 \in \mathtt{SUB}(A)$ and $C_2 \in \mathtt{SUP}(A)$ are never both LFP (Least FixPoint)-concepts or both GFP (Greatest FixPoint)-concepts. Therefore, the subsumption between them cannot be derived using only rules Lfp1, Gfp1, Lfp2, Gfp2. A close look at the above proof system extension reveals that the rules LfpA1, GfpA1, OrLfp are the only rules deriving consequences containing fixpoint concepts from a set of $\mathcal{EL}$ sequents. In all these rules, the sequents before the application of the rule must be cyclic inclusion axioms for a particular atomic concept $A'$. The $\mathcal{EL}$ concept $C(A')$ structurally corresponds to the resulting LFP- or GFP-concept, i.e., the fixpoint concept is obtained from $C(A')$ by replacing all occurrences of $A'$ within $C(A')$ by a concept variable. Moreover, $A' \in N_C \setminus \{A\}$, since $C_1 \in \mathtt{SUB_M}(A)$ and $C_2 \in \mathtt{SUP_M}(A)$. As argued above, no inclusion axioms with an GFP-concept only on the left-hand side or with an LFP-concept only on the right-hand side can be derived unless the TBox contains such inclusion axioms. Since additionally $C(A')$ is an $\mathcal{EL}$ concept, the inclusions $C(A') \sqsubseteq A'$ or $A' \sqsubseteq C(A')$ were derived using only $\mathcal{EL}$ inclusions. For the same reason, the concept $C_1'$ in $\mathtt{SUB}(A')$

such that due to Theorem 8 the inclusion $C(A') \sqsubseteq C'_1$ holds, is an $\mathcal{EL}$ concept. The argumentation for $C'_2 =\in \mathtt{SUP}(A')$ being an $\mathcal{EL}$ concept with $C'_2 \sqsubseteq C(A')$ is analogous. $\square$

**Lemma 3** It remains to prove Lemma 3, which claims that if $\mathtt{SUP}(A)$ for $A \in \overline{\Sigma}_{C,\mathtt{SUP}}$ or $\mathtt{SUB}(A)$ for $A \in \overline{\Sigma}_{C,\mathtt{SUB}}$ together with any concept $C$ is in one set $M \in \mathcal{R}_{\mathtt{MAX}}$, it is contained in all $M \in \mathcal{R}_{\mathtt{MAX}}$. Assume that there exists an $\mathcal{EL}$ TBox $\mathcal{T}'$ such that $\mathcal{T}' \models C_1 \sqsubseteq C_2$ for a non-$\mathcal{EL}$ concept $C_1 \in \mathtt{SUB}_{\mathtt{M}}(A)$, for instance. Assume that $(C_1, C_2) \notin \bigcap \mathcal{R}_{\mathtt{MAX}}$. We know that there is at least one set $M_1 \in \mathcal{R}_{\mathtt{MAX}}$ such that $(C_1, C_2) \in M_1$. Then, there must be a second set $M_2 \in \mathcal{R}_{\mathtt{MAX}}$ such that $(C_1, C_2) \notin M_2$. We know from Lemma 9 that there is a concept $A' \neq A$ and there are two $\mathcal{EL}$ concepts $C'_1 \in \mathtt{SUB}(A')$ and $C'_2 =\in \mathtt{SUP}(A')$ such that $\mathcal{T} \models C_1 \sqsubseteq C'_1$ and $\mathcal{T} \models C'_2 \sqsubseteq C_2$. Assume that $(C'_1, C'_2) \notin M_2$, then it follows that $(C_1, C_2) \in M_2$, since $C'_1 \sqsubseteq C'_2$ is a justification for $C_1 \sqsubseteq C_2$ and the set $M_2$ is maximal, i.e., no axioms can be added to it without loosing consequences in $\mathcal{T}'$. Assume that $(C'_1, C'_2) \in M_2$, then $C'_1$ must be an LFP-concept, which contradicts with the assumption of Lemma 9. To see that $C'_1$ must indeed be an LFP-concept, consider the rules for the extension of $\mathcal{EL}$ with fixpoints. As argued in the proof for Lemma 9, LFP-concepts can only appear on the right-hand side in inclusion axioms, if the concept of the left-hand side of the axiom is also an LFP-concept. $\square$