Generalized Domain-Range Restrictions*

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Abstract. Proposing a certain notion of logical completeness as a novel quality criterion for ontologies, we identify and characterise a class of logical propositions which naturally extend domain and range restrictions commonly known from diverse ontology modelling approaches. We argue for the intuitivity of this kind of axioms and show that they fit equally well into formalisms based on rules as well as ones based on description logics. Extending the attribute exploration technique from formal concept analysis (FCA), we present an algorithm for the efficient interactive specification of all axioms of this form valid in a domain of interest. We compile some results that apply when role hierarchies and symmetric roles come into play and demonstrate the presented method in a small example.

1 Introduction

Ontologies are wonderful. Yet, the practical deployment of semantic technologies in a wider range of applications clearly requires new technical methods as well as methodologies assisting the knowledge engineer in designing medium to large size ontologies containing formalized knowledge beyond the usual subclass-superclass (i.e., taxonomic) relationships.

Though reasoning methods provide some assistance in this regard (e.g., allowing to check for local and global consistency of the formalized knowledge as well as for an ontology's "capability" to logically entail wanted consequences), there are other quality criteria for ontologies that cannot be met by reasoning support alone. One of those central criteria – well-nigh currently neglected in knowledge representation research – is that of *completeness*. More precisely, a knowledge base KB can be said to be complete w.r.t. a certain logic(al fragment), if every statement expressible in that logic can be entailed from KB or declined by KB (e.g. by showing the validity of its negation).

Clearly, completeness w.r.t. expressive formalisms (as, say, OWL1.1-completeness) is a goal which cannot be reasonably fulfilled for non-trivial ontologies. Hence (in analogy to identifying tractable fragments of DLs that allow relatively expressive modelling while still being of low reasoning complexity) we argue for identifying fragments being satisfactorily expressive and intuitive to the user as well as still computationally easy to handle, such that the completeness of a KB w.r.t. those fragments is both desirable and achievable.

Hence in our paper, we characterise a group of axioms which meet those requirements and canonically generalise both domain and range statements. Furthermore we

* This work is supported by the Deutsche Forschungsgemeinschaft (DFG) under the ReaSem project. We furthermore thank Pascal Hitzler and Johanna Völker for their valuable comments.

provide a method for their interactive acquisition that in the end yields a knowledge base being complete w.r.t. the class of these axioms. In Section 2, after some initial motivation, we introduce and define this type of domain axioms expressible equivalently by DL (resp. OWL) statements or by rules. Section 3 presents *Role Exploration*, a method for – given a role (resp. binary predicate) and a set of "interesting" classes (resp. unary predicates) – interactively acquiring all axioms of this type valid in the described domain of interest.[1] This method is based on the attribute exploration algorithm from formal concept analysis. Section 4 discusses how one could take advantage of additional knowledge about roles, namely role hierarchies and role symmetry, by modifications of the Role Exploration algorithm. In Section 5, we demonstrate Role Exploration by further elaborating an example for the setting brought up in Section 2. Finally, Section 6 concludes and gives an outlook to further research.

In the sequel, we assume the reader to be familiar basic notions from description logics (see [1] for a comprehensive and detailed overview) and rule-based languages [2].

2 Generalised Domain-Range Restrictions: Characterisation and Properties

Imagine the following situation: suppose, in a knowledge base describing persons and personal relationships, we have a role denoted with married which is to express whether a person is married to another person. So, clearly an ontology engineer would state that both domain and range of that role would have to be subclasses of Person, being expressed by the DL statements $\exists \mathsf{married}.\top \sqsubseteq \mathsf{Person}$ and $\forall \mathsf{married}.\mathsf{Person}$ or by the rules $\mathsf{married}(X, Y) \rightarrow \mathsf{Person}(X)$ and $\mathsf{married}(X, Y) \rightarrow \mathsf{Person}(Y)$. In an OWL (Web Ontology Language, W3C recommendation [3]) ontology this could be expressed using the domain and range language constructs for object properties as follows:

```
<owl:ObjectProperty rdf:ID="Married">
   <rdfs:domain rdf:resource="#Person"/>
   <rdfs:range rdf:resource="#Person"/>
</owl:ObjectProperty>
```

Yet, what one would certainly like to additionally state is that males can marry only females and vice versa.[2] Obviously, this is not possible via the usual OWL domain and range constructs. However, the DL axioms $\mathsf{Male} \sqsubseteq \forall \mathsf{married}.\mathsf{Female}$ and $\mathsf{Female} \sqsubseteq \forall \mathsf{married}.\mathsf{Male}$ (as well as their OWL DL counterparts) or the rules $\mathsf{married}(X, Y) \wedge \mathsf{Male}(X) \rightarrow \mathsf{Female}(Y)$ and $\mathsf{married}(X, Y) \wedge \mathsf{Female}(X) \rightarrow \mathsf{Male}(Y)$ express exactly this relationship.

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[1] In order not to confuse the two meanings of the term "domain", we use *domain of interest* whenever referring to the meaning "universe of discourse" or "set of all entities".

[2] For the sake of the example we refer to a situation without same-sex marriages. However, this is not meant to reflect any personal attitude of the author towards this topic.

Staying with this kind of examples, note that there are countries (such as India), where the minimal age to get (and hence, to be) married is sex-dependant.[3] The corresponding regulation is no domain or range restriction in the classical sense either, yet can be stated by DL axioms like $\mathsf{Male} \sqcap \exists \mathsf{married}.\top \sqsubseteq \mathsf{Age21plus}$ and $\mathsf{Female} \sqcap \exists \mathsf{married}.\top \sqsubseteq \mathsf{Age18plus}$ or – in a rule language – by $\mathsf{married}(X, Y) \wedge \mathsf{Male}(X) \rightarrow \mathsf{Age21plus}(X)$ and $\mathsf{married}(X, Y) \wedge \mathsf{Female}(X) \rightarrow \mathsf{Age18plus}(Y)$.

Having demonstrated the utility and intuitivity of this kind of modelling axioms, we introduce a type of statements capturing all of them while being still computationally easy to handle.

Definition 1. *Given a set C of named classes and a role R, a* GENERALIZED DOMAIN-RANGE RESTRICTION *(short: GDRR) is a rule having the following form*

$$\mathsf{R}(X, Y) \wedge \bigwedge_{\mathsf{A} \in \mathbf{A}} \mathsf{A}(X) \wedge \bigwedge_{\mathsf{B} \in \mathbf{B}} \mathsf{B}(Y) \rightarrow \bigwedge_{\mathsf{C} \in \mathbf{C}} \mathsf{C}(X) \wedge \bigwedge_{\mathsf{D} \in \mathbf{D}} \mathsf{D}(Y)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \subseteq C$ and R is a role name. Note, that for $\mathbf{C} \cup \mathbf{D} = \emptyset$, the rule will have an empty head (also denoted by $\square$) and, hence, will be interpreted as integrity constraint.

Put into words, the GDRR presented in the above definition would mean the following: "For any two elements X and Y of the domain of interest that are connected by a role R and where X fulfills (all of) $\mathbf{A}$ as well as Y fulfills (all of) $\mathbf{B}$, we know that X additionally fulfills $\mathbf{C}$ and Y additionally fulfills $\mathbf{D}$."

The next theorem guarantees that for every GDRR, there is a semantically equivalent general concept inclusion axiom (GCI) in any sufficiently expressive DL (while these expressiveness requirements are very low).

Theorem 1. *The GDRR*

$$\mathsf{R}(X, Y) \wedge \bigwedge_{\mathsf{A} \in \mathbf{A}} \mathsf{A}(X) \wedge \bigwedge_{\mathsf{B} \in \mathbf{B}} \mathsf{B}(Y) \rightarrow \bigwedge_{\mathsf{C} \in \mathbf{C}} \mathsf{C}(X) \wedge \bigwedge_{\mathsf{D} \in \mathbf{D}} \mathsf{D}(Y)$$

is equivalent to both of the following GCIs:[4]

$$\bigsqcap_{\mathsf{A} \in \mathbf{A}} \mathsf{A} \sqcap \exists \mathsf{R}.\Big(\bigsqcap_{\mathsf{B} \in \mathbf{B}} \mathsf{B}\Big) \sqsubseteq \bigsqcap_{\mathsf{C} \in \mathbf{C}} \mathsf{C} \sqcap \forall \mathsf{R}.\Big(\Big(\bigsqcup_{\mathsf{B} \in \mathbf{B}} \neg \mathsf{B}\Big) \sqcup \Big(\bigsqcap_{\mathsf{D} \in \mathbf{D}} \mathsf{D}\Big)\Big),$$

$$\bigsqcap_{\mathsf{B} \in \mathbf{B}} \mathsf{B} \sqcap \exists \mathsf{R}^{-}.\Big(\bigsqcap_{\mathsf{A} \in \mathbf{A}} \mathsf{A}\Big) \sqsubseteq \bigsqcap_{\mathsf{D} \in \mathbf{D}} \mathsf{D} \sqcap \forall \mathsf{R}^{-}.\Big(\Big(\bigsqcup_{\mathsf{A} \in \mathbf{A}} \neg \mathsf{A}\Big) \sqcup \Big(\bigsqcap_{\mathsf{C} \in \mathbf{C}} \mathsf{C}\Big)\Big).$$

Although the GCI obtained by the uniform translation provided by Theorem 1 might look cumbersome and counterintuitive, note that obviously any GDRR having a conjunction of atoms in the head can be split into several GDRRs with single-atom heads. Each of those will be equivalent to a more intuitive GCI, as stated by the following corollary.

[3] In the Indian *Child Marriage Restraint Act* of 1929, amended in 1978, child is defined as "[...] a person, who, if a male, has not completed twenty-one years of age, and if a female, has not completed eighteen years of age [...]" [4].

[4] where we set $\bigsqcap_{\mathsf{E} \in \mathbf{E}} \mathsf{E}$ to be $\top$ whenever $\mathbf{E} = \emptyset$

Corollary 1. *1. The GDRR of the shape*

$$\mathsf{R}(X,Y) \wedge \mathsf{A}_1(X),\dots,\mathsf{A}_n(X),\mathsf{B}_1(Y),\dots,\mathsf{B}_k(Y) \to \square$$

is equivalent to each of the GCIs

$$\mathsf{A}_1 \sqcap \dots \sqcap \mathsf{A}_n \sqcap \exists\mathsf{R}.(\mathsf{B}_1 \sqcap \dots \sqcap \mathsf{B}_k) \sqsubseteq \bot$$

$$\mathsf{B}_1 \sqcap \dots \sqcap \mathsf{B}_k \sqcap \exists\mathsf{R}^-.(\mathsf{A}_1 \sqcap \dots \sqcap \mathsf{A}_n) \sqsubseteq \bot$$

2. The GDRR of the shape

$$\mathsf{R}(X,Y) \wedge \mathsf{A}_1(X),\dots,\mathsf{A}_n(X),\mathsf{B}_1(Y),\dots,\mathsf{B}_k(Y) \to \mathsf{C}(X)$$

is equivalent to each of the GCIs

$$\mathsf{A}_1 \sqcap \dots \sqcap \mathsf{A}_n \sqcap \exists\mathsf{R}.(\mathsf{B}_1 \sqcap \dots \sqcap \mathsf{B}_k) \sqsubseteq \mathsf{C}$$
$$\mathsf{B}_1 \sqcap \dots \sqcap \mathsf{B}_k \sqsubseteq \forall\mathsf{R}^-.(\neg\mathsf{A}_1 \sqcup \dots \sqcup \neg\mathsf{A}_n \sqcup \mathsf{C})$$
$$\mathsf{B}_1 \sqcap \dots \sqcap \mathsf{B}_k \sqcap \exists\mathsf{R}^-.(\mathsf{A}_1 \sqcap \dots \sqcap \mathsf{A}_n \sqcap \neg\mathsf{C}) \sqsubseteq \bot$$

3. The GDRR of the shape

$$\mathsf{R}(X,Y) \wedge \mathsf{A}_1(X),\dots,\mathsf{A}_n(X),\mathsf{B}_1(Y),\dots,\mathsf{B}_k(Y) \to \mathsf{C}(Y)$$

is equivalent to each of the GCIs

$$\mathsf{A}_1 \sqcap \dots \sqcap \mathsf{A}_n \sqsubseteq \forall\mathsf{R}.(\neg\mathsf{B}_1 \sqcup \dots \sqcup \neg\mathsf{B}_k \sqcup \mathsf{C})$$
$$\mathsf{A}_1 \sqcap \dots \sqcap \mathsf{A}_n \sqcap \exists\mathsf{R}.(\mathsf{B}_1 \sqcap \dots \sqcap \mathsf{B}_k \sqcap \neg\mathsf{C}) \sqsubseteq \bot$$
$$\mathsf{B}_1 \sqcap \dots \sqcap \mathsf{B}_k \sqcap \exists\mathsf{R}^-.(\mathsf{A}_1 \sqcap \dots \sqcap \mathsf{A}_n) \sqsubseteq \mathsf{C}$$

Note that therefore, each of the description logics $\mathcal{ALE}$ and $\mathcal{ELI}$ is sufficient to express GDRRs; for the first two types, even $\mathcal{EL}$ will do.

Considering the rule representation, note that we refrain from using negated atoms. Hence the proposed type of rules belongs to the fragment of Horn clauses. Following the general framework for defining Horn DLs from [5], the DL representation of GDRRs belongs to Horn-$\mathcal{ALE}$ (whereas $\mathcal{ELI}$ is already Horn anyway). Likewise, they also naturally fall in the DLP [6] fragment. Mark that, although no negated atoms are allowed, we can nevertheless express certain kinds of negative statements by using rules with empty heads (also called *integrity constraints*, as mentioned in Definition 1). For example, the statement "a child is not allowed to marry", normally modelled with a DL axiom like $\mathsf{Child} \sqsubseteq \neg\exists\mathsf{married}.\top$, can equivalently be expressed by the GDRR $\mathsf{married}(X,Y), \mathsf{Child}(X) \to \square$.

Hence, GDRRs identify a class of logical statements useful to characterise roles beyond the common domain-range restrictions still being both intuitive and computationally friendly (witnessed by their containment in the abovementioned fragments). Related to that, they also fulfill a certain computationally advantageous locality condition: given the set Δ of all entities of a domain of interest, checking whether a certain GDRR is satisfied therein can be done by separately checking all entity pairs connected by the role R. Mark that this is not the case for any "simple looking" GCI, take for example $\exists\mathsf{has.Sorrow} \sqsubseteq \exists\mathsf{has.Liqueur}$ – a proposition well-known from German poetry.[5]

[5] "Es ist ein Brauch von alters her: *wer Sorgen hat, hat auch Likör!*" (emphasis by the author) to be found in Chapter 16 of [7].

3 Acquisition of GDRRs via Role Exploration

In this section, we will propose a way to exhaustively determine all GDRRs of a certain shape (i.e., referring to a role R and a set of relevant atomic classes C) valid in a domain of interest, i.e., assuring "GDRR-completeness" of the resulting knowledge in the sense introduced in Section 1. This method is based on a well known algorithm from formal concept analysis. The algorithm we present will consequently ask an expert for the validity of GDRRs in the domain of interest and end up with a revised knowledge base and a complete (as defined later) set of GDRRs.

The attribute exploration algorithm our work is based on was introduced in [8]. Attribute exploration with partial or incomplete information has been dealt with in several variants e.g. in [9, 10]. In [11], FCA and DL were combined for the first time by using complex concept descriptions to define new attributes in formal contexts. In [12], attribute exploration was used to determine the concept hierarchy of conjunctions on atomic concepts. The idea to use attribute exploration as a way to interactively refine an ontological knowledge base was brought up in [13] and thoroughly described in [14], where also an extension to the case with partial information was proposed. A concise algorithm for exploration with partly known objects has been provided in [15].

3.1 FCA and Attribute Exploration with Partial Information

We first need to briefly introduce some basic FCA notions. We mainly follow the notation introduced in [16] being *the* reference for FCA theory.

The basic notion FCA is built on is that of a formal context. It is a common claim in FCA that any kind of grounded data can be represented in this way.

Definition 2. *A* FORMAL CONTEXT $\mathbb{K}$ *is a triple* (G, M, I) *with an arbitrary set G (called* OBJECTS*), an arbitrary set M (called* ATTRIBUTES*), and a relation* $I \subseteq G \times M$ *(called* INCIDENCE RELATION*). We read gIm as: "object g has attribute m." Furthermore, let* $g^I := \{m \mid gIm\}$.

For our considerations, we work with a generalised notion of this data structure, allowing for partial specification (i.e., it might be unknown, whether an object has an attribute or not). This is an important extension for a knowledge representation setting, since (due to the OWA), it is reasonable to assume that not all (even not all relevant) facts about a described entity are known.

Definition 3. *A* PARTIAL FORMAL CONTEXT $\mathbb{K}^?$ *is a quadruple* $(G, M, I^\Box, I^\Diamond)$ *where both* $(G, M, I^\Box)$ *and* $(G, M, I^\Diamond)$ *are formal contexts and* $I^\Box \subseteq I^\Diamond$.

A formal context $\mathbb{K} = (G, M, I)$ *will be called* COMPLETION *of* $\mathbb{K}^?$*, if* $I^\Box \subseteq I \subseteq I^\Diamond$.

The intuitive meaning of this definition is the following: $gI^\Box m$ means, it is certain that object g has the attribute m, while $gI^\Diamond m$ means, it is possible that object g has the attribute m or – in other words – it is *not* certain that object g does *not* have the attribute m. An intuitive visualization would be a table with rows corresponding to the objects and columns corresponding to the attributes, having crosses where $gI^\Box m$, blanks where *not* $gI^\Diamond m$ and question marks everywhere else.

Naturally, a completion of a partial formal context will be obtained by substituting each question mark by either a cross or a blank.

In FCA, *implications* constitute the central means of expressing knowledge. We formally specify this rather straightforward notion together with some further useful theory in the following definition.

Definition 4. *Let M be an arbitrary set. An* IMPLICATION *on M is a pair (A, B) with $A, B \subseteq M$. To support intuition, we write $A \rightarrow B$ instead of (A, B).*

$A \rightarrow B$ HOLDS *in a formal context $\mathbb{K} = (G, M, I)$, if for all $g \in G$, we have that $A \subseteq g^I$ implies $B \subseteq g^I$. We then write $\mathbb{K} \models A \rightarrow B$.*

We say, a partial formal context $\mathbb{K} = (G, M, I^\square, I^\Diamond)$ ADMITS *an implication $A \rightarrow B$, if for all $g \in G$ we have that $A \subseteq g^{I^\square}$ implies $B \subseteq g^{I^\Diamond}$. For $C \subseteq M$ and a set $\mathfrak{I}$ of implications on M, let $C^\mathfrak{I}$ denote the smallest set with $C \subseteq C^\mathfrak{I}$ that additionally fulfills*

$$A \subseteq C^I \quad \textit{implies} \quad B \subseteq C^I$$

for every implication $A \rightarrow B$ in $\mathfrak{I}$.[6] *If $C = C^\mathfrak{I}$, we call C $\mathfrak{I}$-*CLOSED. *We say $\mathfrak{I}$* ENTAILS *$A \rightarrow B$ if $B \subseteq A^\mathfrak{I}$.*[7] *An implication set $\mathfrak{I}$ will be called* NON-REDUNDANT, *if for any $(A \rightarrow B) \in \mathfrak{I}$ we have that $B \nsubseteq A^{\mathfrak{I}\setminus\{A\rightarrow B\}}$. A set $\mathfrak{I}$ implications holding in a context $\mathbb{K}$ will be called* COMPLETE, *if every implication $A \rightarrow B$ holding in $\mathbb{K}$ is entailed by $\mathfrak{I}$. $\mathfrak{I}$ will be called an* IMPLICATION BASE *of a formal context $\mathbb{K}$ if it is non-redundant and complete.*

Note that implication entailment is decidable in linear time w.r.t. the size of $\mathfrak{I}$ [17, 18]. Therefore, knowing the implication base in a logical setting allows fast handling of the whole corresponding implicational theory. Moreover, for every formal context, there exists a canonical implication base [19].

The method of attribute exploration allows to acquire the implication base of a domain of interest being just implicitly known by an expert in an interview-like process. Due to space reasons, we omit to display its technical details and refer the reader to the thorough presentation in [15].

Essentially, the following happens: the aspect of the domain of interest that shall be explored is formalized as a formal context $\mathbb{K} = (U, M, I)$. Usually, it is not known completely in advance. However, possibly, some entities of the domain of interest $g \in U$ are already known, as well as some attributes that g has or has not, constituting an initial partial formal context.

During runtime, the algorithm presents questions of the form

"Does the implication $A \rightarrow B$ hold in the context $\mathbb{K} = (U, M, I)$?"

to the human expert. The expert might confirm this. In this case, $A \rightarrow B$ is archived as part of $\mathbb{K}$'s implicational base $\mathfrak{IB}$. The other case would be that $A \rightarrow B$ does not hold in (U, M, I). But then, there must exist a $g \in U$ with $A \in g^I$ and $B \notin g^I$. The expert is asked to input this g and – roughly speaking – enough evidence for qualifying g as a counterexample by augmenting the partial context such that $A \subseteq g^{I^\square}$ and $B \nsubseteq g^{I^\Diamond}$.

[6] Note, that this is well-defined, since the mentioned properties are closed wrt. intersection.

[7] Actually, this is a syntactic shortcut. Yet, it can be easily seen that this coincides with the usual entailment notion.

The procedure terminates when the implicational knowledge of the $\mathbb{K}$ is completely acquired, i.e., the implications admitted by the partial formal context built from the entered counterexamples coincide with those entailed by $\mathfrak{IB}$.

In our approach, we will exploit the capability of attribute exploration to efficiently determine an implicational theory. Notwithstanding, we extend the underlying language[8] from purely propositional to GDRRs.

3.2 Role Contexts

In this work, we employ attribute exploration in a way that is structurally very similar to the approach in [21], where this technique was used for specifying dynamic systems. In this setting, roles would be interpreted as actions that can be taken, classes are used to describe states and the models of a corresponding theory can be interpreted as state transition systems. Yet this technique easily carries over to the more general setting of knowledge specification as firstly sketched by the author in [14].

Definition 5. *Let $\mathcal{KB}$ be a DL knowledge base and, as usual, an interpretation $\mathcal{I}$ of $\mathcal{KB}$ be defined as $(\varDelta, \cdot^{\mathcal{I}})$, where $\varDelta$ is the individual set and $\cdot^{\mathcal{I}}$ a function mapping class names to subsets of $\varDelta$ and role names to subsets of $\varDelta \times \varDelta$.*

For a given interpretation $\mathcal{I}$ together with a set C of named classes and a role R, the ROLE CONTEXT *$\mathbb{K}_{\mathsf{R}}$ is defined as formal context (G, M, I) with*

- $G := \mathsf{R}^{\mathcal{I}} = \{(\delta_1, \delta_2) \mid \delta_1, \delta_2 \in \varDelta, (\delta_1, \delta_2) \in \mathsf{R}^{\mathcal{I}}\}$
  the objects of $\mathbb{K}_{\mathsf{R}}$ are those individual pairs connected by the role R,
- $M := \{\mathsf{C}_{\mathrm{d}}, \mathsf{C}_{\mathrm{r}} \mid \mathsf{C} \in C\}$
  the attribute set of $\mathbb{K}_{\mathsf{R}}$ contains two "copies" of C: the DOMAIN ATTRBUTES *indexed with d the* RANGE ATTRIBUTES *indexed with r, and*
- $I \subseteq G \times M$ *with* $(\delta_1, \delta_2) I \mathsf{C}_{\mathrm{d}} \iff \delta_1 \in \mathsf{C}^{\mathcal{I}}$ *and* $(\delta_1, \delta_2) I \mathsf{C}_{\mathrm{r}} \iff \delta_2 \in \mathsf{C}^{\mathcal{I}}$.
  the domain attributes indicate for an R-connected pair of entities, whether the corresponding class contains the first entity of that pair, while the range attributes describe the second entity.

The following theorem shows how the validity of a GDRR in an interpretation can be read from a corresponding role context.

Theorem 2. *An interpretation $\mathcal{I}$ satisfies a GDRR*

$$\mathsf{R}(X, Y) \wedge \bigwedge_{\mathsf{A} \in \mathbf{A}} \mathsf{A}(X) \wedge \bigwedge_{\mathsf{B} \in \mathbf{B}} \mathsf{B}(Y) \rightarrow \bigwedge_{\mathsf{C} \in \mathbf{C}} \mathsf{C}(X) \wedge \bigwedge_{\mathsf{D} \in \mathbf{D}} \mathsf{D}(Y)$$

if and only if the corresponding role context $\mathbb{K}_{\mathsf{R}}$ satisfies the implication

$$\{\mathsf{A}_{\mathrm{d}} \mid \mathsf{A} \in \mathbf{A}\} \cup \{\mathsf{B}_{\mathrm{r}} \mid \mathsf{B} \in \mathbf{B}\} \rightarrow \bot \quad \textit{if } \mathbf{C} \cup \mathbf{D} = \emptyset \textit{ and}$$

$$\{\mathsf{A}_{\mathrm{d}} \mid \mathsf{A} \in \mathbf{A}\} \cup \{\mathsf{B}_{\mathrm{r}} \mid \mathsf{B} \in \mathbf{B}\} \rightarrow \{\mathsf{C}_{\mathrm{d}} \mid \mathsf{C} \in \mathbf{C}\} \cup \{\mathsf{D}_{\mathrm{r}} \mid \mathsf{D} \in \mathbf{D}\} \quad \textit{otherwise.}$$

[8] There exist already other language extensions, e.g. to Horn-logic with a bounded variable set, see [20].

This theorem enables us to "translate" any implication in a role context into an equivalent GDRR and via Theorem 1 further into a GCI. So, for a given implication $\mathfrak{i}$ from $\mathbb{K}_{\mathsf{R}}$, let $\mathrm{DL}^{+}(\mathfrak{i})$ denote an equivalent GCI with the pure role and $\mathrm{DL}^{-}(\mathfrak{i})$ an equivalent GCI with the inverse role.

Now, the basic idea for the knowledge acquisition method we are going to propose is to carry out attribute exploration (with uncertain knowledge) on the context $\mathbb{K}_{\mathsf{R}}$. Thereby, our basic assumption is that there exists a distinguished interpretation $\mathcal{I}'$ entirely (but implicitly) known by the human expert that we want to specify in terms of GDRRs.

3.3 Reasoner-aided Exploration

The general work flow of exploration based knowledge base refinement was first described by the author in [13] and has been subsequently applied in diverse approaches [22, 14, 15, 23]. Basically, three entities are involved:

- the exploration algorithm consecutively asking questions,
- a reasoner trying to cope with those questions based on (terminological or grounded) information being present a priori (thereby minimising the expert's "workload"), and
- an (ideally omniscient) human expert dealing with those questions that cannot be answered by the reasoner.

For the sake of clarity, we will describe a rather concrete instantiation of this framework. Nevertheless, there are several degrees of freedom in certain parts of the algorithm in that certain additional computation steps could be carried out, which do not alter the outcome of the algorithm but might have significant influence on its performance. We indicate such optional steps in the algorithm leaving questions related to optimisation for future research.

So let $\mathcal{KB}$ be an OWL DL knowledge base and $\mathcal{R}$ be an OWL DL reasoner. Let furthermore $\mathcal{C}$ be a set of named classes and R a role[9] occurring in $\mathcal{KB}$.

Initialisation. We initialise a partial "working" context $\mathbb{K}_R^? = (G, M, I^\square, I^\diamond)$ by setting $G := \emptyset$, $M := \{\mathsf{C}_\mathrm{d}, \mathsf{C}_\mathrm{r} \mid \mathsf{C} \in \mathcal{C}\}$. It will be successively enriched during the exploration.

Scan for a-priori Data (optional). Although any exploration process can be carried out starting from scratch, i.e. without any objects known in advance, such information may be advantageous by making possible hypotheses obsolete. Besides the possibility of manually providing such information, there are two possible ways of extracting this kind of information from a given knowledge base, which we call the *lazy* and the *greedy* way, depending on whether reasoning is employed or not.

So, the lazy way of data search would, for all role statements $\mathsf{R}(a, b) \in \mathcal{KB}$, add (a, b) to the object set G of $\mathbb{K}_R^?$ and set

$$I^\square := I^\square \cup \{((a,b), \mathsf{C}_\mathrm{d}) \mid \mathsf{C}(a) \in \mathcal{KB}, \mathsf{C} \in \mathcal{C}\} \cup \{((a,b), \mathsf{C}_\mathrm{r}) \mid \mathsf{C}(b) \in \mathcal{KB}, \mathsf{C} \in \mathcal{C}\} \text{ and}$$

[9] the corresponding OWL DL term being *object property*

$$\mathcal{I}^{\Diamond} := \mathcal{I}^{\Diamond} \cup \{((a,b), \mathsf{C}_{\mathrm{d}}) \mid \neg\mathsf{C}(a) \notin \mathcal{KB}, \mathsf{C} \in \mathcal{C}\} \cup \{((a,b), \mathsf{C}_{\mathrm{r}}) \mid \neg\mathsf{C}(b) \notin \mathcal{KB}, \mathsf{C} \in \mathcal{C}\}.$$

Clearly, this would just add the relevant information explicitly present in $\mathcal{KB}$ to the working context.

Contrarily, the greedy way would employ reasoning to acquire more complete information to start with. In this case, for any role statement $\mathsf{R}(a,b)$ that can be inferred from $\mathcal{KB}$ by $\mathcal{R}$, the pair (a,b) would be added to G. Employing $\mathcal{R}$ further, we then set

$$\mathcal{I}^{\Box} := \mathcal{I}^{\Box} \cup \{((a,b), \mathsf{C}_{\mathrm{d}}) \mid \mathcal{KB} \models \mathsf{C}(a), \mathsf{C} \in \mathcal{C}\} \cup \{((a,b), \mathsf{C}_{\mathrm{r}}) \mid \mathcal{KB} \models \mathsf{C}(b), \mathsf{C} \in \mathcal{C}\} \text{ and}$$

$$\mathcal{I}^{\Diamond} := \mathcal{I}^{\Diamond} \cup \{((a,b), \mathsf{C}_{\mathrm{d}}) \mid \mathcal{KB} \not\models \neg\mathsf{C}(a), \mathsf{C} \in \mathcal{C}\} \cup \{((a,b), \mathsf{C}_{\mathrm{r}}) \mid \mathcal{KB} \not\models \neg\mathsf{C}(b), \mathsf{C} \in \mathcal{C}\}.$$

Although the greedy way would deliver more starting information which might shorten the subsequent exploration process, this advantage might be vitiated by the large number of possibly time consuming reasoner calls.

Scan for a-priori GDRRs (optional). The exploration algorithms also allows for entering already known implications before starting the actual exploration process. Like in the case with a-priori data, this could accelerate the exploration process, since some hypotheses can be taken for granted.

In order to acquire this kind of information, we check for every GCI occurring in $\mathcal{KB}$, whether it syntactically entails[10] a GDRR (w.r.t. R and $\mathcal{C}$) and if so, add the respective implication $\mathfrak{i}$ to the set of implications known in advance. Note that also GCIs that represent just class hierarchies are interesting in this regard, since e.g. $\mathsf{C} \sqsubseteq \mathsf{D}$ would entail any GDRR $\mathsf{R}(X,Y), \mathsf{C}(X) \to \mathsf{D}(X)$ as well as $\mathsf{R}(X,Y), \mathsf{C}(Y) \to \mathsf{D}(Y)$.

Exploration. Now we start the exploration process on the partial working context. Every hypothetical implication $\mathfrak{i}$ the algorithm comes up with is transformed into a subsumption statement $\mathrm{DL}^+(\mathfrak{i})$. The following two steps can be carried out in arbitrary order (or in parallel), whereas it is impossible that both succeed (which allows to refrain from either one if the other is known to have succeeded).

- Employ $\mathcal{R}$ to check whether $\mathcal{KB} \models \mathrm{DL}^+(\mathfrak{i})$. If so, silently confirm $\mathfrak{i}$ to the exploration algorithm and continue the exploration.
- Employ $\mathcal{R}$ to check whether $\mathcal{KB} \cup \{\mathrm{DL}^+(\mathfrak{i})\}$ is unsatisfiable. If this is the case, this means that $\mathcal{KB}$ forces any model to contain a pair of individuals (i_1, i_2) serving as a counterexample for $\mathfrak{i}$.

If none of the above cases applies, the human expert has to decide whether the proposed GDRR is valid in the described domain of interest, i.e., whether $\mathcal{I}' \models \mathrm{DL}^+(\mathfrak{i})$. If the expert agrees, $\mathfrak{i}$ will be confirmed to the exploration algorithm and additionally – since the expert has revealed genuinely new information – $\mathrm{DL}^+(\mathfrak{i})$ will be added to $\mathcal{KB}$. After that, the exploration continues with a new hypothesis.

In case the GDRR is denied (either by $\mathcal{R}$ or by the expert), a counterexample must be provided. If $\mathcal{R}$ was able to show the unsatisfiability of $\mathcal{KB} \cup \{\mathsf{C} \sqsubseteq \mathsf{D}\}$, it might even

[10] Hereby we mean entailment that can be detected by easy (i.e. tractable) syntactic transformations. Due to lack of space, we postpone an elaboration of this part to future work.

be able to automatically provide a counterexample in the following way. Let $A \rightarrow B$ be the implication in question, and set $\mathcal{G}^+ := \{\mathrm{DL}^+(A \rightarrow \{b\}) \mid b \in B\}$ and $\mathcal{G}^- := \{\mathrm{DL}^-(A \rightarrow \{b\}) \mid b \in B\}$. Now, for every GCI $\mathsf{C} \sqsubseteq \mathsf{D}$ contained in $\mathcal{G}^+ \cup \mathcal{G}^-$, we use $\mathcal{R}$ to retrieve instances of $\mathsf{C} \sqcap \neg\mathsf{D}$. If one such instance, is found, we add a new pair (e_1, e_2) to G and set $I^\square := I^\square \cup \{(e_1, e_2)\} \times A$ as well as $I^\diamond := I^\diamond \cup \{(e_1, e_2)\} \times (M \setminus \{b, \bot\})$. In this case, the exploration process can be continued without consulting the expert.

However, even if the unsatisfiability of $\mathcal{KB} \cup \{\mathsf{C} \sqsubseteq \mathsf{D}\}$ can be shown, there might be no named individual in the $\mathcal{KB}$ witnessing this in the sense just described. Then – as well as in the case when the expert had to deny the hypothetical GDRR himself – he has to manually add information to the knowledge base in a way that a counterexample can be retrieved by the method described above. Obviously, this can be achieved in any case by entering an R-connected individual pair i_1 and i_2 with appropriate class assertions, but there are other ways (as adding instances for one concept description from $\mathcal{G}^+$ or $\mathcal{G}^-$). Then a (partial) counterexample description can be generated automatically in the above described way.

Termination. After the exploration finishes, we have obtained a twofold result:

- A refined version of $\mathcal{KB}$ which is "GDRR-complete" w.r.t $\mathcal{C}$ and R meaning the following: Every GDRR involving the role R and concepts from $\mathcal{C}$ is either entailed by $\mathcal{KB}$ or adding it to $\mathcal{KB}$ leads to unsatisfiability. Hence, $\mathcal{KB}$ completely characterises $\mathcal{I}'$ in terms of this class of GDRRs.
- An implication base $\mathfrak{IB}$, accumulated by the exploration process. $\mathfrak{IB}$ allows to check *in linear time* for *every* GDRR on R and $\mathcal{C}$ whether it is valid in $\mathcal{I}'$ or not.

4 Interplay with other Role Properties

Considering OWL DL, there are lots of other features which can be used to characterise roles. In the sequel we will briefly review how some of this information can be taken advantage of during the role exploration process.

Symmetric Roles. Quite frequently, roles are known to be symmetric. This might be expressed by the DL statement $R \equiv R^-$ or the rule $\mathsf{R}(X, Y) \rightarrow \mathsf{R}(Y, X)$; OWL even provides a dedicated language construct for this. In this case, the symmetry carries over to $\mathbb{K}_\mathsf{R}$ in the following sense: for every implication $A \rightarrow B$ holding in $\mathbb{K}_\mathsf{R}$, the implication $\psi(A) \rightarrow \psi(B)$ with

$$\psi : \left\{ \begin{array}{l} \left.\begin{array}{l} \mathsf{C}_\mathrm{d} \mapsto \mathsf{C}_\mathrm{r} \\ \mathsf{C}_\mathrm{r} \mapsto \mathsf{C}_\mathrm{d} \end{array}\right\} \text{ for all } C \in \mathcal{C} \\ \bot \mapsto \bot \end{array} \right.$$

holds in $\mathbb{K}_\mathsf{R}$ as well. In [24], attribute exploration has been extended in order to take this kind of symmetries into account, allowing the acquisition of implicational knowledge "modulo permutations" on the attribute set.

$\mathsf{Person} \sqsubseteq \mathsf{Male} \sqcup \mathsf{Female}$	$\mathsf{Person} \sqsubseteq \mathsf{Child} \sqcup \mathsf{Adult}$	$\mathsf{Catholic} \sqcap \mathsf{Priest} \sqsubseteq \mathsf{Male}$
$\mathsf{Male} \sqcap \mathsf{Female} \sqsubseteq \bot$	$\mathsf{Child} \sqcap \mathsf{Adult} \sqsubseteq \bot$	$\mathsf{Catholic} \sqcap \mathsf{Protestant} \sqsubseteq \bot$
$\mathsf{married} \equiv \mathsf{married}^-$	$\exists \mathsf{married}.\top \sqsubseteq \mathsf{Person}$	$\top \sqsubseteq \forall \mathsf{married}.\mathsf{Person}$

Fig. 1. Example knowledge base $\mathcal{KB}$ about marriages

Role Hierarchies. A standard feature in expressive description logics (and as well contained in OWL DL) is the definition of role hierarchies. For two given roles $\mathsf{R}_1, \mathsf{R}_2$, the role R_1 is subsumed by the role R_2, (DL notation: $\mathsf{R}_1 \sqsubseteq \mathsf{R}_2$) if $\mathsf{R}_1^{\mathcal{I}} \subseteq \mathsf{R}_2^{\mathcal{I}}$. It takes just little consideration that in this case, every implication valid in $\mathbb{K}_{\mathsf{R}_2}$ is also valid in $\mathbb{K}_{\mathsf{R}_1}$. This can be exploited for the exploration in the following way: Assume for both R_1 and R_2, all valid GDRRs w.r.t. C have to be determined. The most efficient way to do so would then be to first carry out the procedure for R_2 and use the acquired implication base as a-priori knowledge for the next procedure, thereby reducing the amount of hypothetical GDRRs brought up by the algorithm.

5 An Example: So, Who Marries Whom?

For a small demonstration how the presented technique would be applied in practice, let us stay with the example from Section 2. Let $\mathcal{KB}$ be the knowledge base given in Fig. 1. Now imagine, this knowledge base is to be refined with respect to the role $\mathsf{married}$. Let

$$C := \{\mathsf{Person}, \mathsf{Male}, \mathsf{Female}, \mathsf{Child}, \mathsf{Adult}, \mathsf{Catholic}, \mathsf{Protestant}, \mathsf{Priest}\}$$

be the set of interesting class names. So the set of attributes of the role context would be

$$M := \{ \mathsf{Person}_\mathrm{d}, \mathsf{Person}_\mathrm{r}, \mathsf{Male}_\mathrm{d}, \mathsf{Male}_\mathrm{r}, \mathsf{Female}_\mathrm{d}, \mathsf{Female}_\mathrm{r}, \mathsf{Child}_\mathrm{d}, \mathsf{Child}_\mathrm{r}, \mathsf{Adult}_\mathrm{d}, \mathsf{Adult}_\mathrm{r}, \mathsf{Catholic}_\mathrm{d}, \mathsf{Catholic}_\mathrm{r}, \mathsf{Protestant}_\mathrm{d}, \mathsf{Protestant}_\mathrm{r}, \mathsf{Priest}_\mathrm{d}, \mathsf{Priest}_\mathrm{r}, \bot\}$$

Note that the role $\mathsf{married}$ is defined to be symmetric; therefore, the respective additional considerations from the previous section apply. Assume the following married couples already to be known: Andreas & Christiane, Anupriya & Kedar, as well as Astrid & Thomas. So, after initialisation, the starting context would have a shape as depicted in Fig. 2.

In the sequel, we review the hypothetical implications the exploration algorithm comes up with and explain how they are handled by the reasoner and (resp. or) the human expert.

1. Question: $\emptyset \to \{\mathsf{Person}_\mathrm{d}, \mathsf{Adult}_\mathrm{d}, \mathsf{Person}_\mathrm{r}, \mathsf{Adult}_\mathrm{r}\}$ (In words – mark that the empty premise requires the conclusion to be universally true): "If two entities marry, are they both persons and adults?")
   Passing the corresponding GCI (which would be $\exists \mathsf{married}.\top \sqsubseteq \mathsf{Person} \sqcap \mathsf{Adult} \sqcap \forall \mathsf{married}.(\mathsf{Person} \sqcap \mathsf{Adult})$) to the OWL DL reasoner does not yield an answer, since it cannot be derived from the given knowledge base. Hence, the human expert has to be asked and would confirm this implication – since we assume a legal

system where child marriages are prohibited. So the GCI is added to $\mathcal{KB}$ as a new axiom.

2. Question: $\{\mathsf{Male_d}\} \rightarrow \{\mathsf{Female_r}\}$ (In words: "If a male is married, is he necessarily married to a female?")
   This axiom which we already encountered in Section 2 is obviously true but cannot be derived from $\mathcal{KB}$. Therefore, it is passed to the human expert, who again would confirm it which leads to another update of $\mathcal{KB}$
3. Question: $\{\mathsf{Female_d}\} \rightarrow \{\mathsf{Male_r}\}$ (In words: "If a female is married, is she necessarily married to a male?")
   Mark that this axiom is not redundant, since all information specified so far does not exclude the possibility of female-female marriages. Again, the human expert would be asked, confirm the validity and update $\mathcal{KB}$ anew.
4. Question: $\{\mathsf{Female_d}, \mathsf{Male_d}\} \rightarrow \{\bot\}$ (In words: "Is it impossible that somebody married is male and female at the same time?")
   Obviously, the validity of this statement follows from the axiom $\mathsf{Male} \sqcap \mathsf{Female} \sqsubseteq \bot$ contained in the original knowledge base and is therefore silently answered by the reasoner without bothering the expert.
5. Question: $\{\mathsf{Child_d}\} \rightarrow \{\bot\}$ (In words: "Is it impossible for a child to be married?")
   It takes little consideration that this axiom can be derived from the updated knowledge base containing $\mathsf{Child} \sqcap \mathsf{Adult} \sqsubseteq \bot$ as well as the axiom that was added to the $\mathcal{KB}$ as a result of the first question. Thus it is tacitly confirmed by the reasoner as well.
6. Question: $\{\mathsf{Catholic_d}\} \rightarrow \{\bot\}$ (In words: "Is it impossible that a Catholic marries?")
   In fact, since none of the marrying individuals entered so far is Catholic, this is a reasonable hypothesis. Of course it cannot be proved from the current KB, but it cannot be rejected either. Again the expert would have to decide on this. This time, he would decline the hypothesis and enter information witnessing this – possibly a married couple of whom at least one is a Catholic.

In this fashion, the exploration proceeds until it terminates. Only one of the hypotheses presented in the sequel has to be confirmed by the human expert (and consequently added to the knowledge base), namely $\{\mathsf{Catholic_d}, \mathsf{Priest_d}\} \rightarrow \{\bot\}$ – an axiom, the validity of which might become subject to change in the centuries to come.

	$\mathsf{Person_d}$	$\mathsf{Person_r}$	$\mathsf{Male_d}$	$\mathsf{Male_r}$	$\mathsf{Female_d}$	$\mathsf{Female_r}$	$\mathsf{Child_d}$	$\mathsf{Child_r}$	$\mathsf{Adult_d}$	$\mathsf{Adult_r}$	$\mathsf{Catholic_d}$	$\mathsf{Catholic_r}$	$\mathsf{Protestant_d}$	$\mathsf{Protestant_r}$	$\mathsf{Priest_d}$	$\mathsf{Priest_r}$	$\bot$
Andreas & Christiane	×	×	×			×			×	×			×	×			
Anupriya & Kedar	×	×		×	×				×	×							
Astrid & Thomas	×	×		×	×				×	×			×	×	×		

Fig. 2. Starting context for the GDRR-exploration of the role married

6 Conclusion and Future Work

We have motivated and identified a class of OWL axioms that generalise the well-known domain and range restrictions in an intuitive way and can be expressed both in DL-based as well as rule-based formalisms. Moreover, we have proposed an interactive method for refining a knowledge base with respect to a given role (binary predicate) by acquiring all GDRRs valid in a certain domain of interest. As indicated by the given example, we are sure that the proposed technique will be of great help to domain experts and ontology engineers in specifying their domain since it ensures both consistency of the result and completeness in the above described sense.

There are several directions into which we will proceed with our work. An interesting question directly related to the logical fragment of GDRRs is to what extent role involving OWL axioms present in current ontologies can be expressed in the rather restricted form of GDRRs. This would yield an empirical justification for our claim that the identified fragment is of practical interest.

As to the theoretical foundations, an integration of the presented exploration technique with Relational Exploration [14] seems to be promising. Together with the observation, that in recent years, there have been several similar approaches yet differing in the explored logical fragments as well as the additionally used exploration features, the quest for a unifying general theoretical framework would be beneficial since it could both grant theoretical insights as well as spawn versatile joint work towards an integrated implementation which will proof very useful in the context of knowledge specification for the semantic web.

From the perspective of algorithm implementation and optimization, one question longing for empirical clarification is that for the optimal choice of the optional parts of the algorithm, especially, whether “greedy” or “lazy” scan for a-priori information should be applied (this amounts to the question: reasoning whenever possible vs. reasoning only if necessary). Of course, the optimal choice depends on the performance of the reasoner employed w.r.t. the several mentioned reasoning tasks. Since different reasoners might perform differently well in subsumption checking opposed to instance retrieval, it might even be advisable to use several different reasoners.

Finally, the method presented here fits perfectly into recently started work towards a synergetic integration of exploration techniques with complementary approaches from lexical ontology learning aiming at systems that can be beneficially applied in practical situations, as sketched in [23].

In the end, we are very confident, that “completeness-eligible” fragments of common knowledge representation languages in combination with exploration-based techniques will help to establish unprecedented quality standards for ontologies.

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