A Deduction Calculus for Cumulated Clauses on $\mathcal{FLE}$ Concept Descriptions

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Abstract. In this paper, we investigate cumulated clauses on a set of attributes consisting of concept descriptions of the description logic $\mathcal{FLE}$. This kind of expression is useful for describing the attribute logic of contexts where the attributes can be seen as $\mathcal{FLE}$ concept descriptions. We provide a deduction calculus for this type of expressions and prove its soundness and completeness.

1 Introduction

In recent years, several attempts have been made to combine Formal Concept Analysis (FCA) with Description Logic (DL) (see [13, 1, 14, 15, 3, 6]). In particular, DL concept descriptions can be used to to define attributes for formal contexts. This paper is focused on $\mathcal{FLE}$ – a description logic containing conjunction, existential and universal quantification. Cumulated clauses on attributes (logical statements of the shape $\bigwedge A \rightarrow \bigvee_{1 \leq i \leq n} \bigwedge B_i$ with attribute sets $A, B_1, \ldots, B_n$) can be seen as a generalization of attribute implications and are a common way of expressing information in FCA. They have been thoroughly investigated in [11], where also a deduction calculus for "plain" cumulated clauses is given. When considering cumulated clauses on $\mathcal{FLE}$ attributes, it becomes clear that due to the logical interdependencies of $\mathcal{FLE}$ concept descriptions, more derivations on the corresponding cumulated clauses are possible and therefore the deduction calculus has to be revisited in order to keep completeness. In this paper, we present a deduction calculus for this kind of cumulated clauses and prove soundness and completeness thereof.

Such a deduction calculus is valuable in several ways: First, it helps adapting attribute exploration (see [10]) to contexts with $\mathcal{FLE}$ type attributes as sketched in [14] and [15] and concisely worked out in [16]. In this framework, cumulated clauses are used to specify non-implicational background knowledge (see [8]).

Moreover, it can be used to explain automatic inferences to non-sophisticated users (see e.g. [5] for similar work wrt. $\mathcal{ALC}$ subsumption).

2 Description Logic Basics

In general, the term “Description Logic” comprises an amount of different formalisms which vary in expressiveness and decision procedure complexity. This allows to use a certain logic “tailored” to ones needs in practice. For a concise overview for DL, see [2].
In our work, we focus on the well known description logic $\mathcal{FLE}$ which includes conjunction, existential and universal quantification. $\mathcal{FLE}$ concept descriptions are used to describe the entities of a certain domain. Here we recall the definition of the $\mathcal{FLE}$ language.

Definition 1. *Let $M_{\mathcal{C}}$ and $M_{\mathcal{R}}$ be arbitrary finite sets the elements of which we will call* CONCEPT NAMES[1] *and* ROLE NAMES *resp. By $\mathcal{FLE}(M_{\mathcal{C}}, M_{\mathcal{R}})$ (or shortly: $\mathcal{FLE}$ if there is no danger of confusion) we denote the set of concept descriptions being inductively defined as follows:*[2]

$$\begin{aligned} M_{\mathcal{C}} \cup \{\top, \bot\} &\subseteq \mathcal{FLE} \\ \mathtt{C}, \mathtt{D} \in \mathcal{FLE} &\Rightarrow \mathtt{C} \sqcap \mathtt{D} \in \mathcal{FLE} \\ \mathtt{C} \in \mathcal{FLE}, \mathtt{R} \in M_{\mathcal{R}} &\Rightarrow \exists \mathtt{R}.\mathtt{C} \in \mathcal{FLE} \\ \mathtt{C} \in \mathcal{FLE}, \mathtt{R} \in M_{\mathcal{R}} &\Rightarrow \forall \mathtt{R}.\mathtt{C} \in \mathcal{FLE} \end{aligned}$$

Next, we provide a semantics. In order to do this, we first define interpretations. In our case, an interpretation is formalized as binary power context family.

Definition 2. *Let Δ be an arbitrary nonempty set called the* UNIVERSE. *The elements of the universe will be called* ENTITIES. *A* BINARY POWER CONTEXT FAMILY $\overrightarrow{\mathbb{K}}$ *on Δ is a pair $(\mathbb{K}_{\mathcal{C}}, \mathbb{K}_{\mathcal{R}})$ consisting of the formal contexts $\mathbb{K}_{\mathcal{C}} := (G_{\mathcal{C}}, M_{\mathcal{C}}, I_{\mathcal{C}})$ and $\mathbb{K}_{\mathcal{R}} := (G_{\mathcal{R}}, M_{\mathcal{R}}, I_{\mathcal{R}})$ with $G_{\mathcal{C}} = \Delta$ and $G_{\mathcal{R}} = \Delta \times \Delta$.*

Note that our notion of binary power context family is a special case of the power context families as e.g. defined in [17]. There is a canonical one-to-one correspondency to Kripke structures (being used as the usual models for modal logic, see e.g. [4]) or labelled transition systems with attributes (LTSA, see [9]).
Next, we describe an extensional semantics for the above defined concept descriptions: for a given binary power context family $\overrightarrow{\mathbb{K}} = ((\Delta, M_{\mathcal{C}}, I_{\mathcal{C}}), (\Delta \times \Delta, M_{\mathcal{R}}, I_{\mathcal{R}}))$ we assign to each concept description $\mathtt{C} \in \mathcal{FLE}(M_{\mathcal{C}}, M_{\mathcal{R}})$ a set $A \in \mathcal{P}(\Delta)$ of entities[3] that ”fulfill” this concept description.

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[1] Whenever in this publication we use the term *concept*, we refer to the notion used in DL. If we want to refer to the meaning used in FCA, we use *formal concept*.

[2] In DL terminology, it is usual to denote concept descriptions as well as role names by capital letters. In order to avoid possible confusion (with other capital letters denoting sets), we use typewriter font ($\mathtt{A}, \mathtt{B}, \mathtt{C}$) for $\mathcal{FLE}$ concept descriptions and calligraphic letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}$) for concept description sets. Furthermore, we use the symbols $\exists$ and $\forall$ for “role quantification” to clearly distinguish them from the “ordinary” quantifiers $\exists$ and $\forall$ occurring in some proofs and definitions.

[3] Throughout this publication, $\mathcal{P}$ will denote the powerset and $\mathcal{P}_{fin}$ the finite powerset.

Definition 3. *The semantics mapping* $[\![.]\!]_{\overrightarrow{\mathbb{K}}} : \mathcal{FLE}(M_\mathcal{C}, M_\mathcal{R}) \to \mathcal{P}(\Delta)$ *for a binary power context family* $\overrightarrow{\mathbb{K}}$ *on a universe* Δ *with attribute sets* $M_\mathcal{C}$ *and* $M_\mathcal{R}$ *is defined recursively as follows:*

$$\begin{aligned}
[\![\top]\!]_{\overrightarrow{\mathbb{K}}} &:= \Delta \\
[\![\bot]\!]_{\overrightarrow{\mathbb{K}}} &:= \emptyset \\
[\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} &:= \mathtt{C}^{I_\mathcal{C}} \textit{ for all } m \in M_\mathcal{C} \\
[\![\mathtt{C} \sqcap \mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}} &:= [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} \cap [\![\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}} \\
[\![\exists \mathtt{R}.\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} &:= \{\delta_1 \in \Delta \mid \exists \delta_2 \in \Delta : (\delta_1, \delta_2) \in r^{I_\mathcal{R}} \wedge \delta_2 \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\} \textit{ for } \mathtt{R} \in M_\mathcal{R} \\
[\![\forall \mathtt{R}.\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} &:= \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : (\delta_1, \delta_2) \in r^{I_\mathcal{R}} \to \delta_2 \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\} \textit{ for } \mathtt{R} \in M_\mathcal{R}
\end{aligned}$$

For $\delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}$ *we will occasionally write* $\delta \models \mathtt{C}$ *and say* C *is* VALID *in* δ.
Furthermore, a concept description C *is* VALID *in* $\overrightarrow{\mathbb{K}}$ *(which we denote by* $\overrightarrow{\mathbb{K}} \models \mathtt{C}$*), iff* $[\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} = \Delta$. *A concept description* D SUBSUMES *a concept description* C *in* $\overrightarrow{\mathbb{K}}$ *(write:* $\mathtt{C} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{D}$*) iff* $[\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} \subseteq [\![\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}}$. *A concept description* D SUBSUMES *a concept description* C UNIVERSALLY *(write:* $\mathtt{C} \sqsubseteq \mathtt{D}$*) iff* $\mathtt{C} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{D}$ *for all* $\overrightarrow{\mathbb{K}}$ *with attribute sets* $M_\mathcal{C}$ *and* $M_\mathcal{R}$.
Two concept descriptions C *and* D *are called* $\overrightarrow{\mathbb{K}}-$EQUIVALENT *iff* $\mathtt{C} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{D}$ *and* $\mathtt{D} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{C}$ *(write:* $\mathtt{C} \equiv_{\overrightarrow{\mathbb{K}}} \mathtt{D}$*) and* UNIVERSALLY EQUIVALENT *(write:* $\mathtt{C} \equiv \mathtt{D}$*) iff this is the case for all* $\overrightarrow{\mathbb{K}}$ *with attribute sets* $M_\mathcal{C}$ *and* $M_\mathcal{R}$.

It follows directly from the definition of the semantic of $\mathcal{FLE}$ concept descriptions that for any $\mathcal{FLE}$ concept descriptions $\mathtt{C}, \mathtt{D}, \mathtt{E}$ the composed concept descriptions $(\mathtt{C} \sqcap \mathtt{D}) \sqcap \mathtt{E}$ and $\mathtt{C} \sqcap (\mathtt{D} \sqcap \mathtt{E})$ are universally equivalent. The same holds for $\mathtt{C} \sqcap \mathtt{D}$ and $\mathtt{D} \sqcap \mathtt{C}$. In the sequel we will exploit this fact in several ways:

- We will omit all parentheses which are not necessary.
- We will make extensive use of the following abbreviation:
  Let $\mathcal{C} = \{\mathtt{C}_1, \ldots, \mathtt{C}_n\}$ be a finite set of $\mathcal{FLE}$ concept descriptions. Then the new concept description $\mathtt{C}_1 \sqcap \ldots \sqcap \mathtt{C}_n$ will be abbreviated by $\bigsqcap \mathcal{C}$. We extend this definition in an intuitive way for $|\mathcal{C}| < 2$ by setting $\bigsqcap\{\mathtt{C}\} := \mathtt{C}$ and $\bigsqcap \emptyset := \top$. This "syntactic sugar" could then be directly incorporated into the semantic by adding $[\![\bigsqcap \mathcal{C}]\!]_{\overrightarrow{\mathbb{K}}} := \bigcap_{\mathtt{C} \in \mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}$ to Definition 3.
- We will consider all concept descriptions which can be transformed into each other by the equivalences mentioned above as syntactically equivalent, i.e. we write for instance $(\mathtt{C} \sqcap \mathtt{D}) \sqcap \mathtt{E} = (\mathtt{E} \sqcap \mathtt{C}) \sqcap \mathtt{D} = \bigsqcap\{\mathtt{C}, \mathtt{D}, \mathtt{E}\}$.

A notion we will need in the sequel is the MAXIMAL ROLE DEPTH of an $\mathcal{FLE}$ concept description:

Definition 4. *Let* C *be an* $\mathcal{FLE}$*-concept description. The* MAXIMAL ROLE DEPTH *of* C *is given by the function* $\mathfrak{rd} : \mathcal{FLE} \to \mathbb{N}$ *recursively defined as follows:*

$$\begin{aligned}
\mathfrak{rd}(\mathtt{C}) &:= 0 \textit{ for all } \mathtt{C} \in M_\mathcal{C} \cup \{\top, \bot\} \\
\mathfrak{rd}(\mathtt{C} \sqcap \mathtt{D}) &:= \max(\mathfrak{rd}(\mathtt{C}), \mathfrak{rd}(\mathtt{D})) \\
\mathfrak{rd}(\exists \mathtt{R}.\mathtt{C}) &:= \mathfrak{rd}(\mathtt{C}) + 1 \textit{ for all } \mathtt{R} \in M_\mathcal{R} \\
\mathfrak{rd}(\forall \mathtt{R}.\mathtt{C}) &:= \mathfrak{rd}(\mathtt{C}) + 1 \textit{ for all } \mathtt{R} \in M_\mathcal{R}
\end{aligned}$$

For any $n \in \mathbb{N}$ *we define* $\mathcal{FLE}_n := \{\mathtt{C} \mid \mathtt{C} \in \mathcal{FLE}, \mathfrak{rd}(\mathtt{C}) \leq n\}$.

3 $\mathcal{FLE}^{\text{norm}}$ – Reduced, yet Complete

Consider a binary power context family $\vec{\mathbb{K}}$ and let δ be an entity from $\vec{\mathbb{K}}$. Then, knowing that $\mathtt{C}$ and $\mathtt{D}$ are valid in δ, we automatically know that $\mathtt{C} \sqcap \mathtt{D}$ is valid in δ as well. So, one could ask for a "test set" $S \subset \mathcal{FLE}$ such that knowing for every concept description from S whether it is valid in δ allows to conclude this for any $\mathcal{FLE}$ concept description. We will define a concept description set $\mathcal{FLE}^{\text{norm}}$ and show that it has this desired property.

Definition 5. *The set $\mathcal{FLE}^{\text{norm}}$ of* NORMALIZED $\mathcal{FLE}$ CONCEPT DESCRIPTIONS *is an $\mathcal{FLE}$ subset defined in the following way:*

$$M_{\mathcal{C}} \cup \{\bot\} \subseteq \mathcal{FLE}^{\text{norm}}$$
$$\mathcal{C} \in \mathcal{P}_{fin}(\mathcal{FLE}^{\text{norm}}), \bot \notin \mathcal{C}, \mathtt{R} \in M_{\mathcal{R}} \Rightarrow \exists \mathtt{R}.\bigsqcap \mathcal{C} \in \mathcal{FLE}^{\text{norm}}$$
$$\mathtt{C} \in \mathcal{FLE}^{\text{norm}}, \mathtt{R} \in M_{\mathcal{R}} \Rightarrow \forall \mathtt{R}.\mathtt{C} \in \mathcal{FLE}^{\text{norm}}$$

Additionally, for any $i \in \mathbb{N}$, let $\mathcal{FLE}_i^{\text{norm}} = \mathcal{FLE}^{\text{norm}} \cap \mathcal{FLE}_i$.

Theorem 1. *For every $\mathcal{FLE}$-concept description $\mathtt{C}$ there is a set $\mathcal{C}$ of $\mathcal{FLE}^{\text{norm}}$ concept descriptions such that*

$$\mathtt{C} \equiv \bigsqcap \mathcal{C}.$$

Proof sketch: We define a function $\mathfrak{n} : \mathcal{FLE} \to \mathcal{P}(\mathcal{FLE}^{\text{norm}})$ in a recursive manner:

$$\begin{aligned}
\mathfrak{n}(\top) &= \emptyset \\
\mathfrak{n}(\bot) &= \{\bot\} \\
\mathfrak{n}(\mathtt{C}) &= \{\mathtt{C}\} \text{ for } \mathtt{C} \in M_{\mathcal{C}} \\
\mathfrak{n}(\forall \mathtt{R}.\bigsqcap \mathcal{C}) &= \textstyle\bigcup_{\mathtt{C} \in \mathcal{C}} \{\forall \mathtt{R}.\mathtt{D} \mid \mathtt{D} \in \mathfrak{n}(\mathtt{C})\} \\
\mathfrak{n}(\exists \mathtt{R}.\bigsqcap \mathcal{C}) &= \begin{cases} \{\bot\} & \text{if } \bot \in \bigcup_{\mathtt{C} \in \mathcal{C}} \mathfrak{n}(\mathtt{C}) \\ \{\exists \mathtt{R}.\bigsqcap \bigcup_{\mathtt{C} \in \mathcal{C}} \mathfrak{n}(\mathtt{C}) & \text{else.} \end{cases} \\
\mathfrak{n}(\bigsqcap \mathcal{C}) &= \textstyle\bigcup_{\mathtt{C} \in \mathcal{C}} \mathfrak{n}(\mathtt{C}),
\end{aligned}$$

that provides such a set $\mathcal{C}$ for a given $\mathtt{C}$. The fact $\mathtt{C} \equiv \bigsqcap \mathfrak{n}(\mathtt{C})$ can then easily be proven via induction on the maximal role depth of $\mathtt{C}$ (conjunctions can be decomposed into conjunction-free concept descriptions of at most equal maximal role depth). $\square$

Obviously, this theorem provides a way to check on the basis of the "test set" $\mathcal{FLE}^{\text{norm}}$ whether $\delta \models \mathtt{C}$ for any $\mathtt{C} \in \mathcal{FLE}$. This is the case exactly if $\mathfrak{n}(\mathtt{C}) \subseteq \{\mathtt{D} \in \mathcal{FLE}^{\text{norm}} \mid \delta \models \mathtt{D}\}$.

This fact will prove helpful in the next sections since most of our considerations and proofs will have to deal only with $\mathcal{FLE}^{\text{norm}}$ but will propagate to whole $\mathcal{FLE}$.

4 Cumulated Clauses on $\mathcal{FLE}^{\text{norm}}$

Cumulated clauses have been studied and used in FCA as a means of specifying knowledge. In particular, they have been used to encode background knowledge for the attribute exploration process. It can be easily shown that in the case of propositional logic any formula is equivalent to a set of cumulated clauses on the attributes.
Here, we will consider cumulated clauses on $\mathcal{FLE}^{\text{norm}}$. The fact that those attributes have an internal logical structure exerts influence on the clause logic. We will deal with these issues by presenting a sound and complete deduction calculus.
We will start by introducing the notion *cumulated clause*.

Definition 6. *Given an arbitrary set M, a* CUMULATED CLAUSE *on M is an element from $\mathcal{CC}(M) := \mathcal{P}_{\mathit{fin}}(M) \times \mathcal{P}_{\mathit{fin}}\mathcal{P}_{\mathit{fin}}(M)$. To support intuition, we will write $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ instead of $(\mathcal{A}, \{\mathcal{B}_1, \ldots, \mathcal{B}_n\})$ for $\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_n \subseteq M$.*
A set $\mathcal{N} \subseteq M$ is said to RESPECT *a cumulated clause $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ if $\mathcal{A} \not\subseteq \mathcal{N}$ or $\mathcal{B}_i \subseteq \mathcal{N}$ for some $1 \leq i \leq n$.*
A cumulated clause $\mathfrak{k}$ on $\mathcal{FLE}^{\text{norm}}$ is said to be VALID *in a binary power context family $\vec{\mathbb{K}}$ (also: $\vec{\mathbb{K}}$* RESPECTS *$\mathfrak{k}$, written: $\vec{\mathbb{K}} \models \mathfrak{k}$), if for every $\delta \in \Delta$ the set $\{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}\}$ respects it.*
If a cumulated clause $\mathfrak{k}$ is valid in every binary power context family that respects all cumulated clauses from a certain set $\mathfrak{K}$, we say $\mathfrak{k}$ FOLLOWS SEMANTICALLY *from $\mathfrak{K}$ (written $\mathfrak{K} \models \mathfrak{k}$).*

In words, the validity of the cumulated clause $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ in a power context family means that every entity $\delta \in \Delta$ fulfilling all concept descriptions from $\mathcal{A}$ also fulfills all concept descriptions from $\mathcal{B}_1$ or from $\mathcal{B}_2$ ... or from $\mathcal{B}_n$. Obviously, for $n = 1$, the notion of a cumulated clause coincides with that of an implication.

Proposition 1. *A cumulated clause $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ is valid in a binary power context family $\vec{\mathbb{K}}$ iff*

$$\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}}.$$

Proof. We start with the definition of validity and show the equivalence to the statement above:

$$\begin{aligned}
&\forall \delta \in \Delta : \mathcal{A} \not\subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}\} \vee \textstyle\bigvee_{1 \leq i \leq n} \mathcal{B}_i \subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}\} \\
\Longleftrightarrow\ &\forall \delta \in \Delta : \delta \notin \textstyle\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \vee \bigvee_{1 \leq i \leq n} \delta \in \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}} \\
\Longleftrightarrow\ &\forall \delta \in \Delta : \delta \in \textstyle\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \to \delta \in \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}} \\
\Longleftrightarrow\ &\textstyle\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}}.
\end{aligned}$$

□

5 Deduction Calculus on $\mathcal{FLE}^{\text{norm}}$

In the next definition, we present a deduction calculus on $\mathcal{CC}(\mathcal{FLE}^{\text{norm}})$.

Definition 7. *The set $\mathcal{DR}$ of* DERIVATION RULES *consists of the following rules (with $\mathtt{A}, \mathtt{B}, \mathtt{C} \in \mathcal{FLE}^{\text{norm}}$ and $\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}, \mathcal{D}_1, \ldots, \mathcal{D}_k \in \mathcal{P}_{fin}(\mathcal{FLE}^{\text{norm}})$):*

$$\frac{}{\bot \multimap \{\{\mathtt{A}\}\}} \text{ } \textit{contradiction}$$

$$\frac{}{\mathcal{A} \multimap \{\mathcal{A}\}} \text{ } \textit{identity}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}}{\mathcal{A} \cup \{\mathtt{C}\} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}} \text{ } \textit{premise extension}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}}{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}\}} \text{ } \textit{conclusion extension}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}\}, \mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}}{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{D}_1, \ldots, \mathcal{D}_k\}} \text{ } \textit{substitution}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}}{[\exists \mathtt{R}]\mathcal{A} \multimap \{[\exists \mathtt{R}]\mathcal{B}_1, \ldots, [\exists \mathtt{R}]\mathcal{B}_n\}} \text{ } \exists\textit{-lifting}$$

$$\frac{}{[\exists \mathtt{R}]\mathcal{A} \cup \{\forall \mathtt{R}.\mathtt{B}\} \multimap \{[\exists \mathtt{R}](\mathcal{A} \cup \{\mathtt{B}\})\}} \text{ } \forall\textit{-propagation}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}\}}{[\forall \mathtt{R}]\mathcal{A} \multimap \{[\exists \mathtt{R}]\mathcal{B}_1, \ldots, [\exists \mathtt{R}]\mathcal{B}_n, [\forall \mathtt{R}]\mathcal{C}\}} \text{ } \forall\textit{-lifting}$$

where $[\exists \mathtt{R}]\mathcal{A} := \{\exists \mathtt{R}. \bigsqcap \mathcal{A}\}$ and $[\forall \mathtt{R}]\mathcal{A} := \{\forall \mathtt{R}.\mathtt{A} \mid \mathtt{A} \in \mathcal{A}\}$ are used as abbreviations.
Given a set $\mathfrak{K}$ of cumulated clauses, we denote with $\mathcal{DR}(\mathfrak{K})$ the smallest set containing $\mathfrak{K}$ and closed under the derivation rules above. For $\mathfrak{k} \in \mathcal{DR}(\mathfrak{K})$ we also write $\mathfrak{K} \vdash \mathfrak{k}$.

The soundness property of the introduced deduction calculus is stated by the following theorem.

Theorem 2. *For $\mathfrak{K} \subseteq \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$ and $\mathfrak{k} \in \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$:*

$$\mathfrak{K} \vdash \mathfrak{k} \Longrightarrow \mathfrak{K} \models \mathfrak{k}$$

The proof of soundness is rather straightforward: one shows that for every derivation rule

$$\frac{\mathfrak{K}}{\mathfrak{k}} \in \mathcal{DR}$$

with $\mathfrak{K} \subseteq \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$ and $\mathfrak{k} \in \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$ holds that every binary power context family $\overrightarrow{\mathbb{K}}$ respecting all cumulated clauses from $\mathfrak{K}$ also respects $\mathfrak{k}$.
Showing completeness is (as usual) the much more intricate part. We introduce several constructions to achieve that goal.

5.1 The Standard Model

The corresponding proof for $\mathcal{DR}$ will be done in the following way:
Given a set of cumulated clauses on $\mathcal{FLE}^{\mathrm{norm}}$ we will define a particular binary power context family called *standard model*, which

- respects all the given clauses and
- respects just those clauses being derivable from the given ones via $\mathcal{DR}$.

As a consequence, this standard model can serve as a "universal counterexample" against the claim that any non-$\mathcal{DR}$-derivable clause holds in every binary power context family respecting the given clauses.
Note that the usual proof techniques for completeness from modal logic (see e.g. [12]) using maximal consistent formula sets (also known as ultrafilters) is not applicable here, since they require that the considered logic is closed wrt. negation. This is not the case for $\mathcal{FLE}$. For the same reason, adapting other calculi (like that for the multi-modal logic $K_{(m)}$ – see [7]) and corresponding proofs to cumulated clauses on $\mathcal{FLE}$ cannot be easily realized.

Definition 8. *The* STANDARD MODEL $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ *of a given set* $\mathfrak{K} \subset \mathcal{CC}(\mathcal{FLE}^{\mathrm{norm}})$ *is the binary power context family* $\overrightarrow{\mathbb{K}} = (\mathbb{K}_{\mathcal{C}}, \mathbb{K}_{\mathcal{R}}) = ((\Delta, M_{\mathcal{C}}, I_{\mathcal{C}}), (\Delta \times \Delta, M_{\mathcal{R}}, I_{\mathcal{R}}))$ *defined as follows:*

- *First, we set* $\overrightarrow{\mathbb{K}}^{(0)}(\mathfrak{K}) = ((\Delta^{(0)}, M_{\mathcal{C}}, I_{\mathcal{C}}^{(0)}), (\Delta^{(0)} \times \Delta^{(0)}, M_{\mathcal{R}}, I_{\mathcal{R}}^{(0)}))$ *with:*
  - $\Delta^{(0)} := \{\mathcal{N} \subseteq \mathcal{FLE}^{\mathrm{norm}} \mid \mathcal{N}$ *respects all* $\mathfrak{k} \in \mathfrak{K}, \bot \notin \mathcal{N}\}$,
  - $\delta I_{\mathcal{C}}^{(0)} \mathtt{C} :\Leftrightarrow \mathtt{C} \in \delta$, *and*
  - $(\delta_1, \delta_2) I_{\mathcal{R}}^{(0)} \mathtt{R} :\Leftrightarrow \exists\mathtt{R}.\bigsqcap \mathcal{C} \in \delta_1$ *for all finite* $\mathcal{C} \subseteq \delta_2$ *and* $\mathtt{C} \in \delta_2$ *for all* $\forall\mathtt{R}.\mathtt{C} \in \delta_1$.
- *From* $\overrightarrow{\mathbb{K}}^{(n)}(\mathfrak{K})$*, we determine* $\overrightarrow{\mathbb{K}}^{(n+1)}(\mathfrak{K})$ *by*
  - $\Delta^{(n+1)} := \Big\{\delta \in \Delta^{(n)} \mid \{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \delta\} = \bigcap_{(\delta,\widetilde{\delta}) I_{\mathcal{R}}^{(n)} \mathtt{R}} \widetilde{\delta}$ *and* $\{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap \mathcal{C} \in \delta\} = \bigcup_{(\delta,\widetilde{\delta}) I_{\mathcal{R}}^{(n)} \mathtt{R}} \mathcal{P}_{fin}(\widetilde{\delta})\Big\}$
  - $I_{\mathcal{C}}^{(n+1)} := I_{\mathcal{C}}^{(0)} \cap (\Delta^{(n+1)} \times M_{\mathcal{C}})$, *and*
  - $I_{\mathcal{R}}^{(n+1)} := I_{\mathcal{R}}^{(0)} \cap \big((\Delta^{(n+1)} \times \Delta^{(n+1)}) \times M_{\mathcal{R}}\big)$.
- *Finally, we set*
  - $\Delta := \bigcap_{i \in \mathbb{N}} \Delta^{(i)}$,
  - $I_{\mathcal{C}} := I_{\mathcal{C}}^{(0)} \cap (\Delta \times M_{\mathcal{C}})$, *and*
  - $I_{\mathcal{R}} := I_{\mathcal{R}}^{(0)} \cap (\Delta^2 \times M_{\mathcal{R}})$.

Verbally: our standard model is approximated in a (possibly infinite) process, starting by taking as entities all $\mathcal{FLE}^{\mathrm{norm}}$ subsets respecting the given cumulated clauses $\mathfrak{K}$. The implicit aim of the construction is to achieve that every entity fulfills exactly those concept descriptions from $\mathcal{FLE}^{\mathrm{norm}}$ (semantically) that it contains (syntactically). To reach that goal, we successively delete those entities not "compatible" with their "role neighbors". The final standard model can then be seen as the fixed point of this process. In the sequel, we will show that this construction indeed fulfills the intended purpose.

Lemma 1. *Let $\mathfrak{K}$ be a set of cumulated clauses and $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ the corresponding standard model. Then we have for every* $\mathtt{D} \in \mathcal{FLE}^{\text{norm}}$ *and every* $\delta \in \Delta$

$$\mathtt{D} \in \delta \Longleftrightarrow \delta \models \mathtt{D}.$$

Proof. Obviously, for every $\delta \in \Delta$ from $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ holds:

$$\{\mathcal{C} \mid \exists \mathtt{R}.\bigsqcap \mathcal{C} \in \delta\} = \bigcup \{\mathcal{P}_{\mathit{fin}}(\widetilde{\delta}) \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\} \quad (*)$$

as well as

$$\{\mathtt{C} \mid \forall \mathtt{R}.\mathtt{C} \in \delta\} = \bigcap \{\widetilde{\delta} \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\} \quad (**).$$

We do now an induction over the maximal role depth of a concept description $\mathtt{D}$:

- Induction anchor: $\mathtt{D} \in \mathcal{FLE}_0^{\text{norm}}$.
  Then we have either $\mathtt{D} \in M_{\mathcal{C}}$ or $\mathtt{D} = \bot$ In the first case we have $\mathtt{D} \in \delta$ iff $\delta I_{\mathcal{C}} \mathtt{D}$ by definition of the standard model. In view of the semantics definition this is equivalent to $\delta \models \mathtt{D}$.
  Considering the second case, we find that $\bot \in \delta$ does not occur (due to the explicit exclusion of entities containing $\bot$ in the standard model definition). Likewise, $\delta \models \bot$ is never the case since $[\![\bot]\!]_{\overrightarrow{\mathbb{K}}} = \emptyset$. So those both statements are trivially equivalent.
- Induction step: $\mathtt{D} \in \mathcal{FLE}_n^{\text{norm}}$, $n > 0$. Again, we have to distinguish two cases. First, assume $\mathtt{D} = \exists \mathtt{R}.\bigsqcap \mathcal{D}$ with $\mathcal{D} \subseteq \mathcal{FLE}_{n-1}^{\text{norm}}$. Then the statement $\exists \mathtt{R}.\bigsqcap \mathcal{D} \in \delta$ is obviously equivalent to $\mathcal{D} \in \{\mathcal{C} \mid \exists \mathtt{R}.\bigsqcap \mathcal{C} \in \delta\}$ and this because of (*) to $\mathcal{D} \in \bigcup \{\mathcal{P}_{\mathit{fin}}(\widetilde{\delta}) \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\}$. So we know that there exists an R-successor $\widetilde{\delta}$ of δ, which contains all concept descriptions from $\mathcal{D}$. Since $\mathcal{D} \subseteq \mathcal{FLE}_{n-1}^{\text{norm}}$, we see by induction hypothesis that this is the case iff $\widetilde{\delta} \models \mathtt{E}$ for all $\mathtt{E} \in \mathcal{D}$. Subsequently, this is equivalent to $\exists \widetilde{\delta} : (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R} \wedge \widetilde{\delta} \in \bigcap_{\mathtt{E} \in \mathcal{D}} [\![\mathtt{E}]\!]_{\overrightarrow{\mathbb{K}}}$ and this (by the semantics definition) to $\exists \widetilde{\delta} : (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R} \wedge \widetilde{\delta} \in [\![\bigsqcap \mathcal{D}]\!]_{\overrightarrow{\mathbb{K}}}$ and finally to $\delta \in [\![\exists \mathtt{R}.\bigsqcap \mathcal{D}]\!]_{\overrightarrow{\mathbb{K}}}$ which means just $\delta \models \exists \mathtt{R}.\bigsqcap \mathcal{D}$.

  It remains to consider the case $\mathtt{D} = \forall \mathtt{R}.\mathtt{E}$ with $\mathtt{E} \in \mathcal{FLE}_{n-1}^{\text{norm}}$. Then $\forall \mathtt{R}.\mathtt{E} \in \delta$ can be written as $\mathtt{E} \in \{\mathtt{C} \mid \forall \mathtt{R}.\mathtt{C} \in \delta\}$ which is due to (**) equivalent to $\mathtt{E} \in \bigcap \{\widetilde{\delta} \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\}$. Therefore knowing that all R-successors of δ contain E (which is an element of $\mathcal{FLE}_{n-1}^{\text{norm}}$), we conclude by the induction hypothesis that this is equivalent to $\forall \widetilde{\delta} : (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R} \rightarrow \widetilde{\delta} \in [\![\mathtt{E}]\!]_{\overrightarrow{\mathbb{K}}}$ and by the semantics definition to $\delta \in [\![\forall \mathtt{R}.\mathtt{E}]\!]_{\overrightarrow{\mathbb{K}}}$ which means just $\delta \models \forall \mathtt{R}.\mathtt{E}$.

  Note that all argumentations work in both directions. So indeed the equivalence is assured. □

5.2 Realization Trees

Now, we will show that every cumulated clause $\mathfrak{k}$ valid in the standard model can be derived from $\mathfrak{K}$ via $\mathcal{DR}$. This will be done in several steps: First, we define a tree structure that – starting from a given set $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ – represents all "branching possibilities" of extending $\mathcal{A}$ in order to make it respect all cumulated clauses derivable from $\mathfrak{K}$.

Definition 9. *Given a set $\mathfrak{K}$ of cumulated clauses and a set $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$, we call a structure $T^{\mathfrak{K}}_{\mathcal{A}} = (N, r, \prec, \epsilon)$* REALIZATION TREE *of $\mathcal{A}$ if*

- *N is an arbitrary set (the elements of N will be called* NODES*),*
  $r \in N$ (r will also be called the ROOT*),*
  $\prec \subseteq N \times N$ ($\prec$ will be called the SUCCESSOR RELATION*), and*
  $\epsilon : N \to \mathcal{P}(\mathcal{FLE}^{\text{norm}})$,
- *$(N, \prec)$ is a tree with root r,*
- *$\mathcal{A} = \epsilon(r)$,*
- *a node $\nu \in N$ has no successors (i.e., $\nu^{\prec} := \{\widetilde{\nu} \mid \nu \prec \widetilde{\nu}\}$ is empty), if and only if $\epsilon(\nu)$ respects all cumulated clauses from $\mathcal{DR}(\mathfrak{K})$,*
- *otherwise, there exists a cumulated clause $\mathfrak{k} = \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\} \in \mathcal{DR}(\mathfrak{K})$ (called* WITNESSING CLAUSE*) not respected by $\epsilon(\nu)$ with*
  - *$\mathfrak{k}$ is minimal wrt. the greatest role depth occurring in $\mathcal{C}_1, \ldots, \mathcal{C}_n$, and*
  - *among those cumulated clauses fulfilling the conditions above $\mathfrak{k}$'s conclusion is minimal wrt. set inclusion,*

  such that $\nu^{\prec} = \{\nu_1, \ldots, \nu_n\}$ with $\epsilon(\nu_i) = \epsilon(\nu) \cup \mathcal{C}_i$.

Given such a realization tree, we call

- *$(\nu_i)_{i \in \{0,\ldots,k\}}$ a* FINITE COMPLETE PATH *if $\nu_0 = r$, $\nu_i \prec \nu_{i+1}$ for all $0 \leq i < k$ and ν_k has no successors,*
- *$(\nu_i)_{i \in \mathbb{N}}$ an* INFINITE COMPLETE PATH *if $\nu_0 = r$, $\nu_i \prec \nu_{i+1}$ for all $i \in \mathbb{N}$.*
- *$\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ a* LEAF *if $\mathcal{A} = \epsilon(\nu)$ for a $\nu \in \widetilde{N}$ that has no successors,*
- *$\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ a* PSEUDOLEAF *if we have $\mathcal{A} = \bigcup_{i \in \mathbb{N}} \epsilon(\nu_i)$ for some infinite complete path $(n_i)_{i \in \mathbb{N}}$, and*
- *$\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ a* QUASILEAF *if it is a leaf or pseudoleaf.*

Next, we define the term *covering* of a realization tree, being a transversal of all complete paths in this tree.

Definition 10. *Given a realization tree $T = (N, r, \prec, \epsilon)$ a node set $\widetilde{N} \subseteq N$ will be called* COVERING *of T if every (finite or infinite) complete path $r = \nu_0 \prec \nu_1 \prec \ldots$ contains (at least) one element from $\widetilde{N}$.*

Using the fact that a realization tree is finitely branching, it follows immediately from König's Lemma that every arbitrary covering contains a finite one.

Lemma 2. *For every covering $\widetilde{N}$ of a realization tree, there exists a finite covering $N^{\text{fin}} \subseteq \widetilde{N}$.*

In the sequel, we will show that for any cumulated clause "readable" from a realization tree endowed with a covering we can construct a corresponding $\mathcal{DR}$ proof tree.

Lemma 3. *Let $\mathcal{A} \subseteq \mathcal{FLE}^{\mathrm{norm}}$ and let $T_{\mathcal{A}}^{\mathfrak{K}}$ be a realization tree of $\mathcal{A}$. Let furthermore be $\widetilde{N} \subseteq N$ a covering of $T_{\mathcal{A}}^{\mathfrak{K}}$. Let now $\mathcal{C}_1, \ldots, \mathcal{C}_n \subseteq \mathcal{FLE}^{\mathrm{norm}}$ be finite sets such that for every $\nu \in \widetilde{N}$ there is an $i \in \{1 \ldots n\}$ with $\mathcal{C}_i \subseteq \epsilon(\nu)$. Then there is a finite $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathfrak{K} \vdash \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$.*

Proof. W.l.o.g. we can assume $\widetilde{N}$ to be finite due to Lemma 2. We will prove the proposition by showing, that for any node $\nu \in N$ where there is no $\widetilde{\nu} \in \widetilde{N}$ on the path from r to ν there is a finite $\mathcal{B}_\nu \subseteq \epsilon(\nu)$ such that $\mathcal{B}_\nu \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ is $\mathcal{DR}$-derivable. (For $\nu = r$ then follows the claimed result.)
(Note that every ascending path starting from such a ν must contain an element from $\widetilde{N}$, for otherwise we could construct a path starting from r and containing no element from $\widetilde{N}$, which would contradict the precondition.)
Consider all ascending paths starting from such a ν. For every such path $\nu = \nu_0 \prec \nu_1 \prec \ldots$ we can determine the smallest index i such that $\nu_i \in \widetilde{N}$. The greatest one of those smallest indices (there can be only finitely many due to the finiteness of $\widetilde{N}$) will be called the TYPE of ν and denoted by $\tau(\nu)$.
We will prove the proposition by induction over the type of the considered nodes.

- Induction anchor: $\tau(\nu) = 0$
  Then we have $\nu \in \widetilde{N}$ and thus $\mathcal{C}_k \subseteq \epsilon(\nu)$ for some $k \in \{1, \ldots, n\}$. Clearly, $\mathfrak{K} \vdash \mathcal{C}_k \multimap \{\mathcal{C}_k\}$ due to the identity rule. By $(n-1)$fold application of conclusion extension we also find $\mathfrak{K} \vdash \mathcal{C}_k \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. Thus we have found an appropriate $\mathcal{B}_\nu$, namely $\mathcal{B}_\nu := \mathcal{C}_k$.
- Induction step: $\tau(\nu) > 0$
  Then we have $\nu \notin \widetilde{N}$ and all successors $\nu_1, \ldots, \nu_k$ of ν are of type less than $\tau(\nu)$. Thus for every ν_i holds by induction hypothesis that there is a finite $\mathcal{B}_{\nu_i} \subseteq \epsilon(\nu_i)$ with $\mathfrak{K} \vdash \mathcal{B}_{\nu_i} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$.
  Let now be $\mathcal{D} \multimap \{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ the witnessing clause of ν (and therefore in particular derivable). Then we know that $\mathcal{D} \subseteq \epsilon(\nu)$ and $\epsilon(\nu_i) = \epsilon(\nu) \cup \mathcal{E}_i$ for all $i \in \{1, \ldots, k\}$. Now, we can do the following for all ν_i:
  We define $\widetilde{\mathcal{B}}_{\nu_i} := \mathcal{B}_{\nu_i} \cap \epsilon(\nu)$. Then the cumulated clause $\widetilde{\mathcal{B}}_{\nu_i} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ is either equal to $\mathcal{B}_{\nu_i} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ or can be derived from it by applying the premise extension rule. So we know $\mathfrak{K} \vdash \widetilde{\mathcal{B}}_{\nu_i} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. Obviously, $\widetilde{\mathcal{B}}_{\nu_i} \cup \mathcal{D} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\} \vdash \bigcup_{1 \leq j \leq k} \widetilde{\mathcal{B}}_{\nu_j} \cup \mathcal{D} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ can be obtained by several premise extensions (note that all sets are finite, so the union can be realized incrementally). Subsequently, setting $\widetilde{\mathcal{B}} := \bigcup_{1 \leq j \leq k} \widetilde{\mathcal{B}}_{\nu_j}$, we get by substitution $\mathfrak{K} \vdash \widetilde{\mathcal{B}} \cup \mathcal{D} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$.(*)
  Furthermore, knowing $\mathfrak{K} \vdash \mathcal{D} \multimap \{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ we can immediately infer by premise extension $\mathfrak{K} \vdash \widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$.

Together with the clauses from $(*)$ we can do the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{E}_1, ..., \mathcal{E}_k\} \quad \widetilde{\mathcal{B}} \cup \mathcal{D} \cup \mathcal{E}_1 \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n\}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n, \mathcal{E}_2, ..., \mathcal{E}_k\}}\text{MP}}{\vdots}\text{MP}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n, \mathcal{E}_k\}}\text{MP} \quad \widetilde{\mathcal{B}} \cup \mathcal{D} \cup \mathcal{E}_k \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n\}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n\}}\text{MP}$$

So we have $\mathfrak{K} \vdash \widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. But by construction $\widetilde{\mathcal{B}} \cup \mathcal{D}$ is a subset of $\epsilon(\nu)$ and (as a union of finitely many finite sets) also finite. So we can set $\mathcal{B}_\nu := \widetilde{\mathcal{B}} \cup \mathcal{D}$ and we are done. □

In the next lemma, we prove that any quasilieaf of a realization tree respects all clauses from $\mathcal{DR}(\mathfrak{K})$. After this, we show that if $\mathcal{A}$ does not imply $\bot$, none of the realization tree nodes does contain it either.

Lemma 4. *Let $\mathfrak{K}$ be a set of cumulated clauses, $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ and let $T_{\mathcal{A}}^{\mathfrak{K}} = (N, r, \prec, \epsilon)$ be a corresponding realization tree. Then for every quasileaf Q of $T_{\mathcal{A}}^{\mathfrak{K}}$ we have that Q respects all clauses from $\mathcal{DR}(\mathfrak{K})$.*

Proof. Let Q be the quasileaf and $\mathfrak{k} \in \mathcal{DR}(\mathfrak{K})$. We distinguish two cases:

- Q is a leaf with corresponding node ν. Suppose Q does not respect $\mathfrak{k}$. Then either $\mathfrak{k}$ fulfills the minimality conditions from the definition or there is a "smaller" $\widetilde{\mathfrak{k}} \in \mathcal{DR}(\mathfrak{K})$ that does. Thus we have found a possible witnessing clause, which by definition forces ν to have successors. Yet this contradicts our assumption.
- Q is a pseudoleaf with corresponding path $p := (\nu_i)_{i \in \mathbb{N}}$. Suppose Q does not respect $\mathfrak{k}$. Let k be the maximal role depth occurring in $\mathfrak{k}$. Now we set $\widetilde{Q} = Q \cap \mathcal{FLE}_k^{\text{norm}}$. We know that $\widetilde{Q}$ is finite since $\mathcal{FLE}_k^{\text{norm}}$ is finite. Thus, there must exist a node ν_i in p, such that $\widetilde{Q} \in \epsilon(\nu_i)$. Since ν_i is contained in an infinite path, it must have successors. Yet then it has a witnessing clause $\widetilde{\mathfrak{k}} = \mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$. Let $\mathcal{B}_j$ be the set from the conclusion for which $\epsilon(\nu_{i+1}) = \epsilon(\nu_i) \cup \mathcal{B}_j$.
  The maximal role depth of $\mathcal{B}_j$ must be greater than k. Therefore the maximal role depth of $\widetilde{\mathfrak{k}}$'s whole conclusion is greater than k. But then $\widetilde{\mathfrak{k}}$ is not minimal as demanded in the definition, since the maximal role depth of $\mathfrak{k}$'s conclusion is less or equal k and thus definately smaller. So we have found a contradiction to the assumption that there is a $\mathfrak{k} \in \mathcal{DR}(\mathfrak{K})$ not respected by Q. □

Definition 11. *Let $\mathfrak{K}$ be a set of cumulated clauses. A set $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ will be called* CONSISTENT *with respect to $\mathfrak{K}$ if there is no finite set $\mathcal{A}^* \subseteq \mathcal{A}$ such that $\mathfrak{K} \vdash \mathcal{A}^* \multimap \{\{\bot\}\}$.*

Lemma 5. *Let $\mathfrak{K}$ be a set of cumulated clauses and $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ be consistent with respect to $\mathfrak{K}$. For any realization tree $T_{\mathcal{A}}^{\mathfrak{K}} = (N, r, \prec, \epsilon)$ of $\mathcal{A}$ holds that $\epsilon(\nu)$ is consistent for all $\nu \in N$.*

Proof. Assume the contrary. By assumption, we have consistency of $\epsilon(r)$. So if inconsistent nodes ν of $T_{\mathcal{A}}^{\mathfrak{K}}$ exist, there must be some among them, the predecessor $\widetilde{\nu}$ of which is still consistent. Assume ν to be such a minimal inconsistent node. Now let $\mathfrak{k} = \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be the witnessing clause of $\widetilde{\nu}$ and w.l.o.g. $\mathcal{C}_n$ the set with $\epsilon(\nu) = \epsilon(\widetilde{\nu}) \cup \mathcal{C}_n$. Now, due to our assumption we have $\mathfrak{K} \vdash \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ as well as $\mathfrak{K} \vdash \mathcal{D} \cup \mathcal{C}_n \multimap \{\{\bot\}\}$ for a finite $\mathcal{D} \in \epsilon(\widetilde{\nu})$. Then we get by substitution $\mathfrak{K} \vdash \mathcal{D} \cup \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \{\bot\}\}$.
Furthermore, by the contradiction rule and substitution we can easily derive $\mathfrak{K} \vdash \{\bot\} \multimap \{\mathcal{C}_1\}$. Using again substitution it follows $\mathfrak{K} \vdash \mathcal{D} \cup \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}\}$. Yet, the maximal role depth of the conclusion of this new cumulated clause $\widetilde{\mathfrak{k}}$ (the derivability of which has just been shown) is less or equal to that of $\mathfrak{k}$ and furthermore $\widetilde{\mathfrak{k}}$'s conclusion is contained in that of $\mathfrak{k}$. Therefore $\mathfrak{k}$ cannot be the witnessing clause of $\widetilde{\nu}$ since the minimality conditions are violated. So we have a contradiction to the prior assumption. □

5.3 Completeness

Exploiting the two preceding propositions, we now show that any quasileaf of a realization tree with consistent root is an entity of the corresponding standard model. The basic idea of this proof is to show that any such quasileaf "survives" all iterations done in the standard model construction.
Furthermore, we show that any standard model entity has a quasileaf as a subset as well.

Lemma 6. *Let $\mathfrak{K}$ be a set of cumulated clauses and $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ consistent, let $T_{\mathcal{A}}^{\mathfrak{K}} = (N, r, \prec, \epsilon)$ be a corresponding realization tree and $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ the corresponding standard model. Then the following two statements hold:*

1. *for all quasileafs Q of $T_{\mathcal{A}}^{\mathfrak{K}}$, we have $Q \in \Delta$ and*
2. *for all $\delta \in \Delta$ containing $\mathcal{A}$ and being minimal wrt. set inclusion there is a quasileaf Q of $T_{\mathcal{A}}^{\mathfrak{K}}$ with $Q = \delta$.*

Proof. We start with proposition (1) and prove inductively that $Q \in \Delta^{(n)}$ for all $n \in \mathbb{N}$.
Induction anchor: $n = 0$. Obviously, $Q \in \Delta^{(0)}$ for Q respects $\mathcal{DR}(\mathfrak{K})$ (due to Lemma 4) and thus in particular $\mathfrak{K}$. Additionally, we know that Q is consistent (and therefore in particular $\bot \notin Q$) due to Lemma 5.
Induction step: $n > 0$. Considering Q, we have to show that

$$\{\mathcal{C} \mid \exists\mathtt{R}.\mathcal{C} \in Q\} = \bigcup \{\mathcal{P}_{fin}(\widetilde{\delta}) \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\} \qquad (*)$$

and

$$\{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in Q\} = \bigcap \{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\} \qquad (**)$$

We start with (*):

"$\supseteq$" Assume $\mathcal{C}$ to be a subset of an R-successor $\widetilde{\delta}$ of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$. Since by construction we have $I_{\mathcal{R}}^{(n-1)} \subseteq I_{\mathcal{R}}^{(0)}$, we also know that $(Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}$. But in view of the definition of $I_{\mathcal{R}}^{(0)}$, we know that $\exists \mathtt{R}. \bigsqcap \mathcal{C}$ has to be in Q.

"$\subseteq$" By induction hypothesis, we can assume that $Q \in \Delta^{(n-1)}$. Let $\exists \mathtt{R}. \bigsqcap \mathcal{C} \in Q$. We now have to show that (in $\overrightarrow{\mathbb{K}}^{(n-1)}$) there is an R-successor of Q containing $\mathcal{C}$. Suppose there is no such successor. (+)

We set $\widetilde{\mathcal{C}} := \mathcal{C} \cup \{\mathtt{D} \mid \forall \mathtt{R}.\mathtt{D} \in Q\}$ and consider a realization tree $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ (whose quasileafs are all in $\Delta^{(n-1)}$ by induction hypothesis). Due to the assumption (+), no quasileaf of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ is an R-successor of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$ (since each of them contains $\mathcal{C}$). But then (due to the definition of $I_{\mathcal{R}}^{(n-1)}$) no quasileaf of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ is an R-successor of Q in $\overrightarrow{\mathbb{K}}^{(0)}$. So each of these $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$-quasileafs must contradict one of the conditions for being an R-successor of Q in $\overrightarrow{\mathbb{K}}^{(0)}$. Obviously, every quasileaf $\widetilde{Q}$ of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ fulfills the condition that $\mathtt{C} \in \widetilde{Q}$ for all $\forall \mathtt{R}.\mathtt{C} \in Q$, since already $\widetilde{\mathcal{C}}$ contains all such C. So, to fulfill our assumption (+) every $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$-quasileaf $\widetilde{Q}$ must violate the other condition: it has to contain a finite set $\mathcal{D}_{\widetilde{Q}} \in \mathcal{FLE}^{\text{norm}}$ such that $\exists \mathtt{R}. \bigsqcap \mathcal{D}_{\widetilde{Q}} \notin Q$. (++)

For every $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$-quasileaf $\widetilde{Q}$, we find a node ν_p on each of its generating paths p with $\mathcal{D}_{\widetilde{Q}} \subseteq \epsilon(\nu_p)$. Taking for all quasileafs $\widetilde{Q}$ these nodes ν_p, we have found a covering N^* of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$. Due to Lemma 2, we then find also a finite covering $\widetilde{N} \subseteq N^*$. For every $\widetilde{\nu} \in \widetilde{N}$, we choose an arbitrary path $\widetilde{p}$ containing $\widetilde{\nu}$. Let $\widetilde{p}$ generate $\widetilde{Q}$. Now, we again assign a finite $\mathcal{FLE}$ subset $\mathcal{D}_{\widetilde{\nu}}$ to each $\widetilde{\nu}$ by $\mathcal{D}_{\widetilde{\nu}} := \mathcal{D}_{\widetilde{Q}}$. Now let $\{\mathcal{D}_1, \ldots, \mathcal{D}_k\} := \{\mathcal{D}_{\widetilde{\nu}} \mid \widetilde{\nu} \in \widetilde{N}\}$. Using Lemma 3 it follows $\mathfrak{K} \vdash \mathcal{C}^* \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$ for a finite $\mathcal{C}^* \subseteq \widetilde{\mathcal{C}}$. Applying the $\exists$-lifting rule we have also $\mathfrak{K} \vdash [\exists \mathtt{R}]\mathcal{C}^* \multimap \{[\exists \mathtt{R}]\mathcal{D}_1, \ldots, [\exists \mathtt{R}]\mathcal{D}_k\}$.(+ + +)

Furthermore, it is easy to see that $\mathfrak{K} \vdash [\exists \mathtt{R}]\mathcal{C} \cup [\forall \mathtt{R}](\mathcal{C}^* \setminus \mathcal{C}) \multimap \{[\exists \mathtt{R}]\mathcal{C}^*\}$ by incrementally applying the $\forall$-propagation rule (and due to the fact, that $\mathcal{C}^*$ is finite). So together with (+++), using substitution, we finally get

$$\mathfrak{K} \vdash [\exists \mathtt{R}]\mathcal{C} \cup [\forall \mathtt{R}](\mathcal{C}^* \setminus \mathcal{C}) \multimap \{[\exists \mathtt{R}]\mathcal{D}_1, \ldots, [\exists \mathtt{R}]\mathcal{D}_k\}.$$

Due to the construction of $\mathcal{C}$ and $\mathcal{C}^*$, Q contains the premise of this cumulated clause. Furthermore, from Lemma 4 we know that Q has to respect all clauses from $\mathcal{DR}(\mathfrak{K})$. So Q has to contain one element from $\{[\exists \mathtt{R}]\mathcal{D}_1, \ldots, [\exists \mathtt{R}]\mathcal{D}_k\}$ which contradicts the way they have been chosen in (++).

So, our prior assumption (+) must be false.

We continue by showing (**):

"$\subseteq$" Assume C to be a concept description, for which $\forall \mathtt{R}.\mathtt{C} \in Q$. By definition of $I_{\mathcal{R}}^{(0)}$, we know that this implies $\mathtt{C} \in \widetilde{\delta}$ if $(Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}$. So we also know that $\mathtt{C} \in \bigcap \{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}\}$. From $I_{\mathcal{R}}^{(n-1)} \subseteq I_{\mathcal{R}}^{(0)}$, we can conclude that $\bigcap \{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}\} \subseteq \bigcap \{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\}$ and therefore $\mathtt{C} \in \bigcap \{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\}$.

"$\supseteq$" Let $\mathtt{C} \in \bigcap\{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\}$. We have to show that $\forall \mathtt{R.C} \in Q$.
Assume the contrary, i.e., $\forall \mathtt{R.C} \notin Q$. Let $\mathcal{C} := \{\mathtt{D} \mid \forall \mathtt{R.D} \in Q\}$ and consider a realization tree $T_{\mathcal{C}}^{\mathfrak{K}}$ of $\mathcal{C}$ (remember that by induction hypothesis all its quasileafs are in $\Delta^{(n-1)}$). Now we assign to each $T_{\mathcal{C}}^{\mathfrak{K}}$-quasileaf $\widetilde{Q}$ a finite concept description set $\mathcal{D}_{\widetilde{Q}} \subseteq \widetilde{Q}$ in the following way:

- For each quasileaf $\widetilde{Q}$ with $(Q, \widetilde{Q}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}$, we set $\mathcal{D}_{\widetilde{Q}} := \{\mathtt{C}\}$ (this is correct, since $\mathtt{C}$ is contained in every $\mathtt{R}$-successor of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$).
- If a quasileaf $\widetilde{Q}$ is not an $\mathtt{R}$-successor of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$, it cannot be an $\mathtt{R}$-successor of Q in $\overrightarrow{\mathbb{K}}^{(0)}$ as well. Yet, then it must violate one of the two conditions in the definition of $I_{\mathcal{R}}^{(0)}$. Obviously, every quasileaf $\widetilde{Q}$ of $T_{\mathcal{C}}^{\mathfrak{K}}$ fulfills the condition that $\mathtt{C} \in \widetilde{Q}$ for all $\forall \mathtt{R.C} \in Q$, since $\mathcal{C}$ contains all such $\mathtt{C}$. So the second condition must be violated and thus there has to be a finite concept description set $\mathcal{E} \subseteq \widetilde{Q}$ with $\exists \mathtt{R}. \bigcap \mathcal{E} \notin Q$. Then we set $\mathcal{D}_{\widetilde{Q}} := \mathcal{E}$.

Now, since all those assigned concept description sets are finite, we find on every generating path p of a quasileaf $\widetilde{Q}$ a node ν_p for which already holds $\mathcal{D}_{\widetilde{Q}} \subseteq \epsilon(\nu_p)$. Collecting all those nodes, we get a covering N^* of $T_{\mathcal{C}}^{\mathfrak{K}}$. Due to Lemma 2, we find a finite covering $\widetilde{N} \subseteq N^*$. For every $\widetilde{\nu} \in \widetilde{N}$, we choose an arbitrary path $\widetilde{p}$ containing $\widetilde{\nu}$. Let $\widetilde{p}$ generate $\widetilde{Q}$. We again assign a finite $\mathcal{FLE}$ subset $\mathcal{D}_{\widetilde{\nu}}$ to each $\widetilde{\nu}$ by $\mathcal{D}_{\widetilde{\nu}} := \mathcal{D}_{\widetilde{Q}}$ Now let $\{\mathcal{D}_1, \ldots, \mathcal{D}_k\} := \{\mathcal{D}_{\widetilde{\nu}} \mid \widetilde{\nu} \in \widetilde{N}\}$. By using Lemma 3 it follows $\mathfrak{K} \vdash \mathcal{C}^* \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$ for a finite $\mathcal{C}^* \subseteq \mathcal{C}$. If $\{\mathtt{C}\}$ is not yet contained in $\{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$, we may easily include it by one application of the conclusion extension rule. So we get $\mathfrak{K} \vdash \mathcal{C}^* \multimap \{\{\mathtt{C}\}, \mathcal{E}_1, \ldots, \mathcal{E}_j\}$ with $\exists \mathtt{R}. \bigcap \mathcal{E}_i \notin Q$ (as the $\mathcal{E}_i$ have been chosen). But now a single application of the ($\forall$-lifting) rule yields

$$\mathfrak{K} \vdash [\forall \mathtt{R}]\mathcal{C}^* \multimap \{\{\forall \mathtt{R.C}\}, [\exists \mathtt{R}]\mathcal{E}_1, \ldots, [\exists \mathtt{R}]\mathcal{E}_j\}.$$

Since Q as a quasileaf of $T_{\mathcal{A}}^{\mathfrak{K}}$ has to respect all cumulated clauses of $\mathcal{DR}(\mathfrak{K})$ (due to Lemma 4) and by construction $[\forall \mathtt{R}]\mathcal{C}^* \subseteq Q$ it has to contain either $\forall \mathtt{R.C}$ (which contradicts our first assumption) or one $[\exists \mathtt{R}]\mathcal{E}_i$ which contradicts the choice of the $\mathcal{E}_i$. So our assumption $\forall \mathtt{R.C} \notin Q$ must be false.

We proceed with proposition (2). Since we know that $\epsilon(r) = \mathcal{A}$, we also know $\epsilon(r) \subseteq \delta$.
Now we construct a path $r = \nu_0 \prec \nu_1 \prec \ldots$ in $T_{\mathcal{A}}^{\mathfrak{K}}$ in the following way: If ν_i has no successors, we are done and have constructed a complete finite path. Otherwise we select the node ν_{i+1} as follows: We presuppose that for a ν_i we have $\epsilon(\nu_i) \subseteq \delta$. Considering the witnessing clause $\mathfrak{k} = \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of ν_i in $T_{\mathcal{A}}^{\mathfrak{K}}$, we know that δ must respect $\mathfrak{k}$ due to the soundness of $\mathcal{DR}$. Furthermore, by Lemma 1, we have the correspondence of (syntactic) containment and (semantic) validity of $\mathcal{FLE}^{\text{norm}}$ concept descriptions in the standard model. So, since the premise of the witnessing clause is contained in $\epsilon(\nu_i)$ which in turn is a subset of δ, we have $\delta \models \mathcal{B}$ and therefore we have $\delta \models \mathcal{C}_k$ for some $k \in \{1, \ldots, n\}$. Now we choose ν_{i+1} such that $\epsilon(\nu_{i+1}) = \epsilon(\nu_i) \cup \mathcal{C}_k$, thereby assuring $\epsilon(\nu_{i+1}) \subseteq \delta$.

Now let Q be the quasileaf generated by the (finite or infinite) complete path $\nu_0 \prec \nu_1 \prec \ldots$. Due to the first part of the theorem we know that $Q \in \Delta$. By construction, we also know that $\mathcal{A} \subseteq Q$ as well as $Q \subseteq \delta$. However, since δ is minimal wrt. set inclusion by assumption we can conclude $Q = \delta$. □

Having established this correspondence between standard model and realization tree, it is not difficult to prove that any cumulated clause valid in the standard model is derivable via $\mathcal{DR}$, which (as the subsequent corollary shows) gives us the completeness of $\mathcal{DR}$.

Theorem 3. *Let $\mathfrak{K}$ be a set of cumulated clauses and $\mathfrak{k} = \mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ a cumulated clause. Then*

$$\overrightarrow{\mathbb{K}}(\mathfrak{K}) \models \mathfrak{k} \Longrightarrow \mathfrak{K} \vdash \mathfrak{k}$$

Proof. Consider a realization tree $T_{\mathcal{A}}^{\mathfrak{K}}$ of $\mathcal{A}$. From Theorem 6, we know that for each quasileaf Q of $T_{\mathcal{A}}^{\mathfrak{K}}$ holds $Q \in \Delta$. From $\mathcal{A} \subseteq Q$ and using Lemma 4, we can conclude $\mathcal{B}_i \subseteq Q$ for some $i \in \{1, \ldots, n\}$.
Since all $\mathcal{B}_i$ are finite, we find on every complete path a node ν for which already holds $\mathcal{B}_i \subseteq \epsilon(\nu)$ for some $\mathcal{B}_i$. This means that we have found a covering N^* of $T_{\mathcal{A}}^{\mathfrak{K}}$, that due to Lemma 2 can be minimized to a finite covering $\widetilde{N} \subseteq N^*$. In view of Lemma 3 we then get $\mathfrak{K} \vdash \mathcal{A}^* \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ for some $\mathcal{A}^* \subseteq \mathcal{A}$ and consequently by premise extension $\mathfrak{K} \vdash \mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$. □

Corollary 1. *The deduction calculus $\mathcal{DR}$ for cumulated clauses on $\mathcal{FLE}^{\text{norm}}$ is sound and complete.*

Proof. Soundness has been shown by Theorem 2. Completeness also follows directly from the preceding theorem: If a cumulated clause $\mathfrak{k}$ is valid in all power context families that respect a set $\mathfrak{K}$ of cumulated clauses, it is in particular valid in $\overrightarrow{\mathbb{K}}(\mathfrak{K})$. But then it is derivable. □

6 Conclusion and Outlook

In this paper, we presented a sound and complete deduction calculus for cumulated clauses on $\mathcal{FLE}$ concept descriptions. This "logic of case distiction" will be useful for the application of FCA attribute exploration in domains where attributes are specifiable by $\mathcal{FLE}$ concept descriptions. Additionally, we want to emphasize that the results and notions introduced in this paper can also be used in a larger framework. For instance, based on a restricted standard model, a decision algorithm for entailment on cumulated clauses on $\mathcal{FLE}$ can be defined, which differs from the well known tableaux based algorithms in many aspects. These issues (as well as additional structural properties of the standard model) are thoroughly addressed in [16].

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