Four-Valued Semantics for Default Logic *

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Abstract. Reiter's default logic suffers the triviality, that is, a single contradiction in the premise of a default theory leads to the only trivial extension which everything follows from. In this paper, we propose a default logic based on four-valued semantics, which endows default logic with the ability of handling inconsistency without leading to triviality. We define four-valued models for default theory such that the default logic has the ability of nonmonotonic paraconsistent reasoning. By transforming default rules in propositional language $\mathcal{L}$ into language $\overline{\mathcal{L}}^+$, a one-to-one relation between the four-valued models in $\mathcal{L}$ and the extensions in $\overline{\mathcal{L}}^+$ is proved, whereby the proof theory of Reiter's default logic is remained.

1 Introduction

Reiter's default logic [1] is an important nonmonotonic logic. It has been studied widely for its clarity in syntax as well as strong power in reasoning. In the default logic, a set of formulae W and a set of default rules D form a default theory (W, D). Reiter's default logic is supposed to reason in consistent knowledge: even a single contradiction presented in W will lead to the unique trivial extension which includes everything.

One way to make default logic handle inconsistent knowledge is to resolve the contradictions in the premise of a default theory. The signed system [2] decomposes the connection between positive atoms and negative ones by formulae transformation and then restores a consistent set of formulae by default logic. The set of formulae transformed from the original one is consistent and it is used as W in a default theory. It follows that all extensions are nontrivial. Roughly speaking, the signed system does not aim at handling inconsistencies in a nonmonotonic logic, since the default rules are not used in knowledge representation. In the bi-default logic [3], all parts of default rules are transformed in a same way, and then default theories are transformed into bi-default theories. Because of the differences between its proof theory and that of default logic, it will take much effort to implement the bi-default logic.

The systems listed above have a similar character, that is they provide procedures of two steps: transforming and then computing. However, they lack semantics. Also it is hard to point out the direct connections between the inconsistent default theory and its conclusions.

Some nonmonotonic paraconsistent logics (see [4, 5] among others) have been proposed by directly introducing nonmonotonicity into paraconsistent logics, especially

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Belnap's four-valued logic [6, 7]. However, in the computing level, there are challenges in implementing effective theorem provers for them.

Our main contribution in this paper is to provide four-valued semantics for default logic whereby we gain a nonmonotonic paraconsistent logic, named four-valued default logic, in which we can reason under a nice semantics but by a classical proof theory. So the semantics works as an interface of nonmonotonic paraconsistent reasoning, and the procedure of transforming and computing just serves as a tool to compute the models of default theories. This novel reasoning method makes the four-valued default logic applicable in commonsense reasoning. Inheriting the proof theory of Reiter's default logic and equipped with semantics of Belnap's four-valued logic, our four-valued default logic is a paraconsistent version of the former and a nonmonotonic extension of the latter.

We develop our work in the following steps. First of all, four-valued models are defined as semantics of default logic. As we know, an extension of a default theory is a minimal set satisfying both W and D in the context expressed by the extension itself. We adopt the similar approach. A four-valued model of a default theory is minimal in the sense of "information" and satisfies both W and D in the context expressed by the model itself. Similarly, our method can be extended to any other minimalities.

Secondly, we propose a uniform procedure to compute four-valued models in the context of Reiter's default logic. A transformation of default rules is provided with which default theories in $\mathcal{L}$ are transformed into those in $\overline{\mathcal{L}}^+$, and then the computation of models is converted to that of extensions. Why we can do so is that we have gotten the one-to-one relation between the four-valued models in $\mathcal{L}$ and the extensions in $\overline{\mathcal{L}}^+$. Consequently, the four-valued semantics for default logic can be easily implemented by classical reasoning systems for the original default logic [8].

The logic DL3 presented in [9] also combines default logic with a multi-valued logic, Lukasiewicz three-valued logic. But DL3 also suffers the triviality. On the other hand, the proof theory of Reiter's default logic is modified in DL3 unlike that of the four-valued default logic. Comparison details appear in section 6.

By briefly reviewing Reiter's default logic and Belnap's four valued logic in section 2 and 3 respectively, we focus on the k-minimal model in section 4, its computation and the transformation of default theories in section 5. Finally, we compare our work with some others in section 6 and conclude this paper in Section 7.

2 Default Logic

Let $\mathcal{L}$ be a propositional language. A theory is a set of formulae, and $Th(\cdot)$ denotes the consequence operator on propositional logic.

A *default theory* is a pair (W, D), where W is a theory of $\mathcal{L}$ and D is a set of *default rules* of the form: $\frac{\alpha:\beta}{\gamma}$. The formulae α, β, γ of $\mathcal{L}$ are called *prerequisite*, *justification* and *conclusion* respectively. For the sake of simplicity, we assume that there is one and only one justification in a default rule, and this restriction is not essential (see [10]). We denote the prerequisite, justification and conclusion of a default rule δ as $Preq(\delta)$, $Jus(\delta)$ and $Cons(\delta)$ respectively. A default theory may have none, a single or multiple *extensions* defined by a fixed point:

Definition 1 ([1]). *Let $T = (W, D)$ be a default theory. For any set S of formulae, let $\Gamma(S)$ be the minimal set that satisfies:*

(D1) $\Gamma(S) = Th(\Gamma(S))$*;*
(D2) $W \subseteq \Gamma(S)$*;*
(D3) $if\ \frac{\alpha:\beta}{\gamma} \in D, \alpha \in \Gamma(S)\ and\ \neg\beta \notin S, then\ \gamma \in \Gamma(S)$.

A set E is an extension of T iff $\Gamma(E) = E$.

Any extension represents a possible belief set expressed by the default theory.
Reiter's default logic can not deal with inconsistencies:

Proposition 1 ([1]). *A default theory $T = (W, D)$ has an inconsistent extension iff W is inconsistent, and it is the only extension of T.*

3 Four-Valued Logic

As four truth-values in Belnap's logic [6, 7, 5], $FOUR = \{t, f, \top, \bot\}$ (also written as $(1, 0), (0, 1), (1, 1)$ and $(0, 0)$ respectively) intuitively represent truth, falsity, inconsistency and lack of information respectively. The four truth-values form a *bilattice* $(FOUR, \leq_t, \leq_k)$ named $\mathcal{FOUR}$, where the partial orders are defined as the following rules: for every $x_1, x_2, y_1, y_2 \in \{0, 1\}$,

$$(x_1, y_1) \leq_t (x_2, y_2) \text{ iff } x_1 \leq x_2\ and\ y_1 \geq y_2;$$
$$(x_1, y_1) \leq_k (x_2, y_2) \text{ iff } x_1 \leq x_2\ and\ y_1 \leq y_2.$$

Intuitively, the partial order $\leq_t$ reflects differences in the amount of *truth*, while $\leq_k$ reflects differences in the amount of *information*. The first element x of the truth-value pair (x, y) stands for a formula and the second against it.

It follows the operators of $\mathcal{FOUR}$: $\neg(x, y) = (y, x)$, $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2)$, $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2)$, $(x_1, y_1) \supset (x_2, y_2) = (\neg x_1 \vee x_2, x_1 \wedge y_2)$, and $(x_1, y_1) \rightarrow (x_2, y_2) =_{df} \neg(x_1, y_1) \vee (x_2, y_2)$.

In four-valued logic, *internal implication* is interpreted as operator $\supset$ and *material implication* is interpreted as operator $\rightarrow$ in $\mathcal{FOUR}$. We use the same symbols to denote connectives in $\mathcal{L}$ and operators on $\mathcal{FOUR}$.

A *four-valued valuation* v is a function that assigns a truth value from $FOUR$ to each atom in $\mathcal{L}$. Any valuation is extended to complex formulae in the obvious way. A valuation v is a *four-valued model* of (or *satisfies*) a formula ψ if $v(\psi) \in \{t, \top\}$.

Definition 2 ([5]). *Let Σ be a set of formulae and ψ a formula in $\mathcal{L}$. Denote $\Sigma \models^4 \psi$, if every four-valued model of Σ is a four-valued model of ψ.*

Let v and u be four-valued valuations, denote $v \leq_k u$ if $v(p) \leq_k u(p)$ for every atom p. Given a formulae set Σ in $\mathcal{L}$, the minimal elements w.r.t. $\leq_k$ in all models of Σ are called the *k-minimal models* of Σ.

Definition 3 ([5]). *Let Σ be a set of formulae and ψ a formula in $\mathcal{L}$, Denote $\Sigma \models^4_k \psi$ if every k-minimal model of Σ is a model of ψ.*

4 Four-Valued Default Logic

Let $\mathcal{L}$ be a propositional language that does not contain constants t, f, $\top$ and $\bot$. All logic connectives in $\mathcal{L}$ are $\neg, \vee, \wedge, \rightarrow$ and $\supset$, where $\rightarrow$ is defined by $\neg$ and $\vee$ in the usual way. Suppose Δ is a set of models, we denote $\Delta(\phi) \in truthSet$ if $\forall M \in \Delta, M(\phi) \in truthSet$, where $truthSet$ is a subset of $FOUR$ and ϕ is a formula in $\mathcal{L}$.

In Reiter's default logic, a single (classical) model cannot represent beliefs. One of the reasons is that a single model cannot differentiate "being false" and "not being true". By deductive closed theory, which is equal to a set of (classical) models, we can say that ϕ is false if $\neg\phi$ is in that theory, and that ϕ is not known (i.e. "not being true" and "not being false") if both ϕ and $\neg\phi$ are not in the theory. In the case of four-valued logics, we can distinguish them by non-classical truth values. So, we can use one single four-valued model to represent beliefs expressed by default theories.

A default theory may have none, a single or multiple k-minimal models defined by:

Definition 4. *Let $T = (W, D)$ be a default theory in $\mathcal{L}$. For any four-valued valuation N on $\mathcal{L}$, let $\Gamma_k(N)$ be the biggest set of four-valued valuations on $\mathcal{L}$ satisfying that:*

(Ax) *If $N' \in \Gamma_k(N)$ then N' is a four-valued model of W.*
(K-min) *If $N' \in \Gamma_k(N)$ then $N' \leq_k N$.*
(Def) *If $\frac{\alpha:\beta}{\gamma} \in D$, $\Gamma_k(N)(\alpha) \in \{t, \top\}$ and $N(\beta) \in \{t, \bot\}$, then $\Gamma_k(N)(\gamma) \in \{t, \top\}$.*

A valuation M is a k-minimal model of T iff $\Gamma_k(M) = \{M\}$.

A singleton is required in the condition $\Gamma_k(M) = \{M\}$, because any other model in the set $\Gamma_k(M)$ includes less information than the context M does and it should be eliminated when reconstructing the context. The condition K-min indicates that all information achieved should be restricted by the context.

Definition 5. *We say a model M' satisfies a default theory T in the context of M, if*

- *M' is a four-valued model of W, and*
- *If $\frac{\alpha:\beta}{\gamma} \in D$, $M'(\alpha) \in \{t, \top\}$ and $M(\beta) \in \{t, \bot\}$, then $M'(\gamma) \in \{t, \top\}$.*

It is easy to show that M satisfies T in the context of M itself, if M is a k-minimal model of $T = (W, D)$, and what's more, M is the $\leq_k$-minimal one:

Theorem 1. *If M is a k-minimal model of a default theory $T = (W, D)$, then M is a $\leq_k$-minimal model that satisfies T in the context of M.*

Example 1. Let $W = \{p, \neg p\}$, $D = \{\frac{p:r}{q}\}$, $T = (W, D)$. W has four models that assign r the value $\bot$: $M_1(q) = t$, $M_2(q) = f$, $M_3(q) = \top$, $M_4(q) = \bot$ and they all assign p the value $\top$. If N is a model of W and $N(r) \neq \bot$, there exists a model $M_i \in \Gamma_k(N), 1 \leq i \leq 4$. Since $\Gamma_k(M_1) = \{M_1\}$, $\Gamma_k(M_2) = \Gamma_k(M_4) = \emptyset$, and $\Gamma_k(M_3) = \{M_1, M_2, M_3, M_4\}$, M_1 is the only one k-minimal model of T.

Example 2. Let $W = \{p, \neg p\}$, $D_1 = \{\frac{p:p\wedge r}{q}\}$. $T_1 = (W, D_1)$ has only one k-minimal model M_1 s.t. $M_1(p) = \top, M_1(q) = \bot, M_1(r) = \bot$.

Let $D_2 = \{\frac{p:r}{q}, \frac{p\wedge\neg p:r}{q\wedge\neg q}\}$. $T_2 = (W, D_2)$ has only one k-minimal model M_2 s.t. $M_2(p) = \top, M_2(q) = \top, M_2(r) = \bot$.

Notice that, when we replace W by the set $\{p\}$, each default theory in the above examples has only one k-minimal model M s.t. $M(p) = t, M(q) = t, M(r) = \bot$.

Some contradictions introduced by default rules can also be handled "properly":

Example 3. Let $T = (\emptyset, \{\frac{:p}{q}, \frac{:p}{\neg q}\})$. T lacks extensions, while T has one k-minimal model: $M(p) = \bot, M(q) = \top$.

Example 4 (Tweety dilemma). A representation in four-valued logic is given as following (see [5]):

$$W_0 = \begin{cases} bird\_Tweety \rightarrow fly\_Tweety \\ penguin\_Tweety \supset bird\_Tweety \\ penguin\_Tweety \supset \neg fly\_Tweety \end{cases}$$

$W = W_0 \cup \{bird\_Tweety\}, W' = W_0 \cup \{penguin\_Tweety\}$.

The k-minimal four-valued models of W and W' are shown in Table 1.

Table 1. k-minimal models of W and W'

		$bird\_Tweety$	fly_Tweety	$penguin\_Tweety$
W	M_1	t	t	$\bot$
	M_2	$\top$	$\bot$	$\bot$
W'	M_3	$\top$	f	t
	M_4	t	$\top$	t

When all we know about Tweety is that it is a bird, we can not draw the reasonable conclusion that Tweety can fly by four-valued logic (in its k-minimal reasoning). When knowing more about Tweety that Tweety is a penguin, we are confused with whether Tweety is a bird (for we have the negative knowledge that Tweety is not a bird).

In the four-valued default logic, we can get an alternative representation:

$$T_0 = \begin{cases} p \wedge \neg p \\ penguin\_Tweety \supset bird\_Tweety \\ penguin\_Tweety \supset \neg fly\_Tweety \\ bird\_Tweety : fly\_Tweety / fly\_Tweety \end{cases}$$

where $(p \wedge \neg p)$ stands for any contradiction. Denote $T_0 = (W_1, D)$, $W_2 = W_1 \cup \{bird\_Tweety\}, T = (W_2, D)$ and $W' = W_2 \cup \{penguin\_Tweety\}, T' = (W', D)$.

The k-minimal models of T and T' are shown in Table 2.

Table 2. k-minimal models of T and T'

		$bird\_Tweety$	fly_Tweety	$penguin\_Tweety$	p
T	M_1'	t	t	$\bot$	$\top$
T'	M_2'	t	f	t	$\top$

Just as expected, when what we know about Tweety is only that it is a bird, we think it can fly. After knowing that Tweety is a special bird: a penguin, we revise our beliefs and claim that it can't fly without being confused.

In example 4, because of the presence of contradictions in p, Reiter's default logic will collapse, but in the four-valued default logic, the inconsistencies are successfully localized and do not do any harm to reason about Tweety.

Definition 6. *Let $T = (W, D)$ be a default theory and ϕ be a formula in $\mathcal{L}$. Denote $T \models^k \phi$, if for any k-minimal model M of T, $M(\phi) \in \{t, \top\}$ holds.*

Theorem 2. *$W \models_k^4 \phi$ iff $(W, \emptyset) \models^k \phi$.*

Theorem 2 shows that the four-valued default logic in its k-minimal reasoning pattern can be viewed as an extension of four-valued logic in k-minimal reasoning. And as a consequence, only the skeptical consequence relation (defined in definition 6) is suitable for the four-valued default logic.

The next theorem provides a more intutive characterization of k-minimal models of a default theory.

Theorem 3. *If $T = (W, D)$ is a default theory in $\mathcal{L}$, then a four-valued valuation M is a k-minimal four-valued model of T iff $\bigcap_{i=0}^{\infty} \overline{M}_i = \{M\}$, where*

$$\overline{M}_0 = \{N \leq_k M \mid N \text{ is a four-valued model of } W\}$$
$$\overline{M}_{i+1} = \{N \in \overline{M}_i \mid N(\gamma) \in \{t, \top\}, \frac{\alpha : \beta}{\gamma} \in D,$$
$$\text{where } \overline{M}_i(\alpha) \in \{t, \top\} \text{ and } M(\beta) \in \{t, \bot\}\}$$

The four-valued default logic has some nice properties shown in the followings.

Definition 7. *Suppose $T = (W, D)$ is a default theory and M is a k-minimal model of T. The set of* generating defaults *for M w.r.t. T is defined to be $GD(M, T) = \{\frac{\alpha:\beta}{\gamma} \in D \mid M(\alpha) \in \{t, \top\}, M(\beta) \in \{t, \bot\}\}$.*

Theorem 4. *Suppose $T = (W, D)$ is a default theory. If M is a k-minimal model of T then M is a k-minimal model of $(W \cup Cons(GD(M, T)))$.*

Theorem 5 (k-minimality). *Suppose that M and N are k-minimal models of a default theory $T = (W, D)$, where $Jus(D)$ does not include the internal implication $\supset$. If $M \leq_k N$ then M and N are identical.*

Theorem 5 indicates that sometimes we need restrict the occurrences of internal implication in $Jus(D)$ to achieve nice properties, but we also need internal implication in W, $Preq(D)$, and $Cons(D)$ to strengthen the expressive power.

5 Computing k-Minimal Models of Default Theories

Let $\overline{\mathcal{L}}^+$ be the objective language of formulae transformation satisfying that $\mathcal{L} \cap \overline{\mathcal{L}}^+ = \emptyset$ and $\mathcal{A}(\overline{\mathcal{L}}^+) = \{p^+, p^- \mid p \in \mathcal{A}(\mathcal{L})\}$, where operator $\mathcal{A}(\mathcal{L})$ denotes all atoms in $\mathcal{L}$. And $\overline{\mathcal{L}}^+$ only includes logic connectives: $\neg$, $\vee$, $\wedge$ and $\rightarrow$. Notice that the internal implication and the material implication coincide in the classical logic.

5.1 Transformation of Formulae

In [11, 12, 2, 3], the technique of transformation has been proved very useful. In this subsection, we show this method in a convenient way.

Definition 8. *For every formula ϕ in $\mathcal{L}$, $\overline{\phi}^+$ in $\overline{\mathcal{L}}^+$ is a transformation of ϕ if:*

1. $\overline{\phi}^+ = p^+$, *where* $\phi = p, p \in \mathcal{A}(\mathcal{L})$
2. $\overline{\phi}^+ = p^-$, *where* $\phi = \neg p, p \in \mathcal{A}(\mathcal{L})$
3. $\overline{\phi}^+ = \overline{\varphi}^+ \vee \overline{\psi}^+$, *where* $\phi = \varphi \vee \psi$
4. $\overline{\phi}^+ = \overline{\varphi}^+ \wedge \overline{\psi}^+$, *where* $\phi = \varphi \wedge \psi$
5. $\overline{\phi}^+ = \neg\overline{\varphi}^+ \vee \overline{\psi}^+$, *where* $\phi = \varphi \supset \psi$
6. $\overline{\phi}^+ = \overline{\psi}^+$, *where* $\phi = \neg\neg\psi$
7. $\overline{\phi}^+ = \overline{\neg\varphi}^+ \wedge \overline{\neg\psi}^+$, *where* $\phi = \neg(\varphi \vee \psi)$
8. $\overline{\phi}^+ = \overline{\neg\varphi}^+ \vee \overline{\neg\psi}^+$, *where* $\phi = \neg(\varphi \wedge \psi)$
9. $\overline{\phi}^+ = \overline{\varphi}^+ \wedge \overline{\neg\psi}^+$, *where* $\phi = \neg(\varphi \supset \psi)$

In the rest of the paper, we denote $\overline{\Sigma}^+ = \{\overline{\phi}^+ \mid \phi \in \Sigma\}$.

Theorem 6. *$\overline{W}^+$ is (classical) consistent for any theory W.*

We call a theory E complete if it contains p or $\neg p$ for every atom $p \in \mathcal{A}(E)$.

Definition 9. *Let E^+ be a theory in $\overline{\mathcal{L}}^+$,define a map v_{E^+} on $\mathcal{L}$ w.r.t. E^+ as:*

$$v_{E^+}(\phi) = \begin{cases} \top = (1,1) & \overline{\phi}^+ \in E^+, \overline{\neg\phi}^+ \in E^+ \\ t = (1,0) & \overline{\phi}^+ \in E^+, \neg\overline{\neg\phi}^+ \in E^+ \\ f = (0,1) & \neg\overline{\phi}^+ \in E^+, \overline{\neg\phi}^+ \in E^+ \\ \bot = (0,0) & \neg\overline{\phi}^+ \in E^+, \neg\overline{\neg\phi}^+ \in E^+ \end{cases}$$

Obviously, the map v_{E^+} is a valuation when E^+ is consistent and complete.

Theorem 7. *If E^+ is a consistent and complete theory in $\overline{\mathcal{L}}^+$, then the map v_{E^+} is a four-valued valuation on $\mathcal{L}$, i.e.: $v_{E^+}(\neg\phi) = \neg v_{E^+}(\phi)$, $v_{E^+}(\phi \vee \psi) = v_{E^+}(\phi) \vee v_{E^+}(\psi)$, $v_{E^+}(\phi \wedge \psi) = v_{E^+}(\phi) \wedge v_{E^+}(\psi)$, and $v_{E^+}(\phi \supset \psi) = v_{E^+}(\phi) \supset v_{E^+}(\psi)$.*

Definition 10. *Let v be a valuation on $\mathcal{L}$, define the complete and deductive closed theory E_v^+ w.r.t. v in $\overline{\mathcal{L}}^+$ by:*

$$\begin{aligned} E_v^+ = Th\big(&\{p^+ \mid p \in \mathcal{L}, v(p) \in \{t, \top\}\} \cup \{p^- \mid p \in \mathcal{L}, v(p) \in \{f, \top\}\} \\ \cup\ &\{\neg p^+ \mid p \in \mathcal{L}, v(p) \in \{f, \bot\}\} \cup \{\neg p^- \mid p \in \mathcal{L}, v(p) \in \{t, \bot\}\}\big) \end{aligned}$$

Proposition 2. *The theory E_v^+ w.r.t. v is (classical) consistent.*

Theorem 8. *Let E_v^+ be the theory w.r.t. a given valuation v, then*

1. $\overline{\phi}^+ \in E_v^+$ if $v(\phi) \in \{t, \top\}$; $\overline{\neg\phi}^+ \in E_v^+$ if $v(\phi) \in \{f, \top\}$.
2. $\neg\overline{\phi}^+ \in E_v^+$ if $v(\phi) \in \{f, \bot\}$; $\neg\overline{\neg\phi}^+ \in E_v^+$ if $v(\phi) \in \{t, \bot\}$.

Thus the correspondence between consistent and complete deductive closed theories and four-valued valuations is built up completely.

Corollary 1. *Let v be a valuation on $\mathcal{L}$ and E^+ be a consistent complete and deductive closed theory, then v is w.r.t. E^+ iff E^+ is w.r.t. v.*

5.2 Relation Between Models and Extensions

From definition 9, we can see that under the transformation it is reasonable to declare ϕ is true (or false) when $\overline{\phi}^+$ (or $\overline{\neg\phi}^+$) is present, while the presence of $\neg\overline{\phi}^+$ (or $\neg\overline{\neg\phi}^+$) states the lack of information of "being true (or false)". In the sense of information keeping, transformation is naturally extended to commit default rules:

Definition 11. $\mathcal{T}(D) = \{\overline{\alpha}^+ : \neg\overline{\neg\beta}^+ / \overline{\gamma}^+ \mid \alpha : \beta/\gamma \in D\}$

In definition 11, the prerequisite and the justification are transformed in such different ways that we can easily distinguish different beliefs they stand for.

In order to minimize the statements drawn by the default theory, we explicitly import $\neg p^+$ and $\neg p^-$ by default to declare that we lack the information about whether p is true and false respectively.

Definition 12. $D^k = \{\frac{:\neg p^+}{\neg p^+}, \frac{:\neg p^-}{\neg p^-} \mid p \in \mathcal{A}(\mathcal{L})\}$.

Definition 13. *The k-minimal transformation of default theory T is defined by*

$$\mathcal{T}^k(T) = (\overline{W}^+, \mathcal{T}(D) \cup D^k).$$

Theorem 9. *All extensions of $\mathcal{T}^k(T)$ are consistent and complete.*

The following example shows how our technique of transformation works:

Example 5. Suppose that $T = (\emptyset, \{\frac{:p}{q}, \frac{:p}{\neg q}\})$. T has no extensions, while $\mathcal{T}^k(T)$ has a unique extension: $E^+ = Th(\{\neg p^+, \neg p^-, q^+, q^-\})$. Let M be the model w.r.t. E^+, then M is just the only one k-minimal model of T, as shown in example 3.

Theorem 10. *Let M be a four-valued model in $\mathcal{L}$, $\overline{M}_i$ is defined as in theorem 3. Then E^+ w.r.t. M is an extension of $\mathcal{T}^k(T)$ iff $E^+ = \bigcup_{i=0}^{\infty} E_i^+$, where $E_i^+ = \bigcap_{N \in \overline{M}_i} E_N^+$, and E_N^+ is the theory w.r.t. N.*

Theorem 11. *Let M be a k-minimal model of a default theory $T = (W, D)$. If E^+ is the theory w.r.t. M, then E^+ is an extension of $\mathcal{T}^k(T)$.*

Theorem 12. *Let $T = (W, D)$ be a default theory in $\mathcal{L}$ and E^+ is an extension of $\mathcal{T}^k(T)$. If M is the valuation w.r.t. E^+, then M is a k-minimal model of T.*

Corollary 2. *Let T be a default theory in $\mathcal{L}$ and $\phi \in \mathcal{L}$. Then $T \models^k \phi$ iff $\overline{\phi}^+$ is in every extension of $\mathcal{T}^k(T)$.*

Thus we can get four-valued models of a default theory by computing extensions of its counterpart transformed from itself and vise versa as shown in Fig. 1.

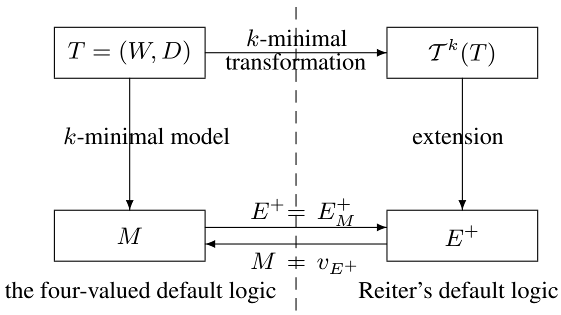

Fig. 1. The relationship between the four-valued default logic and Reiter's default logic

6 Related Work

The default logic in the signed system [2] is only used to restore contents from formulae in $\overline{\mathcal{L}}^+$, which are transformed from the original ones in $\mathcal{L}$. In this paper, we presented a paraconsistent variant of default logic.

In terms of proposing a variant and an extension of Reiter's default logic, one of the previous work is the bi-default logic [3]. The bi-default logic (or Reiter's default logic) is incomparable to the four-valued default logic in reasoning power. For example, the default theory in example 3 has a k-minimal four-valued model but lacks bi-extensions (and extensions). But when there is no default rules present, the four-valued default logic may infer less conclusions than the bi-default logic (or default logic) does, which is based on classical logic. Secondly, although a map from the bi-extensions to $FOUR$ is given, we can not get four-valued models of a default theory. In fact, the map is even not a four-valued valuation, e.g. there is a map which gives both ϕ and ψ the same value $\top$ but assigns $\phi \wedge \psi$ the value f. But we explicitly defined four-valued models for default theory. Finally, in the four-valued default logic, the prerequisite and the justification of a default rule are transformed into different forms, unlike the case of the bi-default logic, in which the bi-extension is defined to justify whether a default rule is applicable. One advantage of our method is that the proof theory of Reiter's default logic is preserved.

In the method proposed in [11, 12], circumscription is used as a tool to calculate multi-valued preferential models in classical logic. But circumscription is weaker than default logic [13], so their method is also weaker than ours in expressive and reasoning power.

The three-valued default logic DL3 [9] is based on Lukasiewicz three-valued logic LUK3. By introducing modal like operators M and L, a formula can be declared to be "possibly" true or "certainly" true in DL3. Since LUK3 is not paraconsistent [14], DL3 also collapses whenever the premise is not consistent. Considering the adopted approaches, there are two main differences between DL3 and the four-valued default logic. First, we defined four-valued models for default theory instead of extensions done in DL3. Second, we can get all four-valued models of every default theory by computing extensions in standard default logic. But Radzikowska only discussed the proof theory limited to normal default theories in the original paper [9], by simulating that of Reiter's default logic.

7 Conclusion

In this paper, we proposed the k-minimal four-valued semantics for default theory. As an extension of Belnap's four-valued logic [6, 7, 5] and a paraconsistent version of Reiter's default logic [1], the four-valued default logic can handle inconsistencies and it still uses default theories in knowledge representation.

A novel technique was also provided to transform default theories into the ones without trivial extensions. The one-to-one correspondence between the extensions of default theory gained by transformation and the four-valued models of the original one was set up as shown in Fig.1. Thus, four-valued models of default theory can be computed by default logic theorem provers (e.g. [8]).

In this paper, we defined k-minimal models for default theory, and we confirmed that our method can be applied to other minimalities. The results of this paper are limited to propositional level, we will extend it to first-order case, as well as consider the applications of the four-valued default logic in commonsense reasoning in the future.

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