Terminological Reasoning in $\mathcal{SHIQ}$ with Ordered Binary Decision Diagrams

Technical Report

Sebastian Rudolph and Markus Krötzsch and Pascal Hitzler

Institut AIFB, Universität Karlsruhe, Germany
{sru,mak,phi}@aifb.uni-karlsruhe.de

Abstract. We present a new algorithm for reasoning in the description logic $\mathcal{SHIQ}$, which is the most prominent fragment of the Web Ontology Language OWL. The algorithm is based on ordered binary decision diagrams (OBDDs) as a datastructure for storing and operating on large model representations. We thus draw on the success and the proven scalability of OBDD-based systems. To the best of our knowledge, we present the very first algorithm for using OBDDs for reasoning with general Tboxes.

1 Introduction

In order to leverage intelligent applications for the Semantic Web, scalable reasoning systems for the standardised Web Ontology Language OWL[1] are required. OWL is essentially based on description logics (DLs), with the DL known as $\mathcal{SHIQ}$ currently being its most prominent fragment.

State-of-the art OWL reasoners, such as Pellet,[2] RacerPro[3] or KAON2[4] already achieve an efficiency which makes them suitable for practical use, however they still fall short of the scalability requirements needed for large-scale applications. The prominent reasoners are essentially based on two differing approaches to reasoning with DLs: While systems such as Pellet and RacerPro are based on tableau algorithms, KAON2 uses a resolution-based approach. The development of such fundamentally different reasoning approaches has furthered the progress in scalable OWL reasoning substantially, both by means of cross-fertilisation between the different systems, and by showing that different algorithms perform differently depending on the knowledge bases and the reasoning tasks [2].

In this paper, we present a new promising algorithm for reasoning with $\mathcal{SHIQ}$, which is based on ordered binary decision diagrams (OBDDs) as a datastructure for storing and operating on large model representations [3–5]. The rationale behind the approach is the fact that OBDD-based systems feature impressive efficiency on large amounts of data, e.g. for model checking for hard- and software verification [6]. Our algorithm is

[1] `http://www.w3.org/2004/OWL/`, see also [1].
[2] `http://pellet.owldl.com/`
[3] `http://www.racer-systems.com/de/index.phtml?lang`
[4] `http://kaon2.semanticweb.org/`

indeed based on a reduction of $\mathcal{SHIQ}$ reasoning to standard OBDD-algorithms, and thus allows to draw on the available strong algorithms and implementations for OBDDs, such as JavaBDD[5].

The general idea of using OBDDs for reasoning with description logics is not entirely new, and some related results have already been presented in [7]. Indeed, a closer look reveals that certain temporal logics to which OBDDs have been applied (e.g. CTL [5]) are closely related to modal logics which in turn are known to have strong structural similarities to DLs [8]. Hence, it seems almost natural to apply OBDD-based techniques for DL reasoning as well. The results from [7], however, are still rather restricted since they encompass only terminological reasoning in the basic DL $\mathcal{ALC}$ without general Tboxes.

In essence, OBDDs can be used to represent arbitrary Boolean functions. These Boolean functions are then interpreted as a kind of compressed encoding of – usually very large sets of – process states. Model checking and certain manipulations of the state space can then be done directly on this compressed version without unfolding it. In our approach, we will employ OBDDs in a very similar way for encoding DL interpretations. However, as DL reasoning is concerned with all possible models, we will show by model-theoretic arguments that for our purposes it is sufficient to work only with certain representative models.

A birds eyes' persepective on our results is as follows: $\mathcal{SHIQ}$ knowledge bases can be reduced equisatisfiably to $\mathcal{ALCI}b$ knowledge bases (Section 5). A sound and complete decision procedure based on so-called domino interpretations provides the next step (Section 3). This procedure can in turn be realised by manipulating Boolean functions (Section 4), which establishes the link with OBDD-algorithms.

We have chosen to present the material in a somewhat different order as it should make the paper more accessible: Preliminaries are given in Section 2. Then in Section 3 we establish model theoretic results for the description logic $\mathcal{ALCI}b$, provide the decision procedure and show that it is sound and complete. In Section 4, we establish the link with operations on Boolean functions. Section 5 provides and justifies a way of transforming a knowledge base in the DL $\mathcal{SHIQ}$ into an equisatisfiable $\mathcal{ALCI}b$ knowledge base. Finally, we conclude and give an outlook to future work in Sections 6 and 7.

2 Preliminaries

In this section we will introduce some auxiliary constructs and propositions as well as all the basic DL notions needed in this paper.

2.1 The Description Logic $\mathcal{SHIQ}b$

We start by recalling some basic definitions of DLs (see [9] for a comprehensive treatment of DLs) and introducing our notation. We define a rather expressive description logic $\mathcal{SHIQ}b$ that extends $\mathcal{SHIQ}$ with restricted Boolean role expressions [10]. We will not consider $\mathcal{SHIQ}b$ knowledge bases, but the DL serves as a convenient umbrella

[5] `http://javabdd.sourceforge.net`

logic for the DLs used in this paper. Also, we do not consider assertional knowledge, and hence will only introduce terminological axioms here.

Definition 1. *A terminological $\mathcal{SHIQ}b$ knowledge base is based on two disjoint sets of* concept names N_C *and* role names N_R. *A set of* atomic roles $\mathbf{R}$ *is defined as* $\mathbf{R} := \mathsf{N}_R \cup \{R^- \mid R \in \mathsf{N}_R\}$. *In addition, we set* $\mathrm{Inv}(R) := R^-$ *and* $\mathrm{Inv}(R^-) := R$, *and we will extend this notation also to sets of atomic roles. In the sequel, we will use the symbols* R, S *to denote atomic roles, if not specified otherwise.*
The set of Boolean role expressions $\mathbf{B}$ *is defined as follows:*

$$\mathbf{B} ::= \mathbf{R} \mid \neg\mathbf{B} \mid \mathbf{B} \sqcap \mathbf{B} \mid \mathbf{B} \sqcup \mathbf{B}.$$

We use $\vdash$ *to denote standard Boolean entailment between sets of atomic roles and role expressions. Given a set* $\mathcal{R}$ *of atomic roles, we inductively define:*

- *For atomic roles* R, $\mathcal{R} \vdash R$ *if* $R \in \mathcal{R}$, *and* $\mathcal{R} \nvdash R$ *otherwise,*
- $\mathcal{R} \vdash \neg U$ *if* $\mathcal{R} \nvdash U$, *and* $\mathcal{R} \nvdash \neg U$ *otherwise,*
- $\mathcal{R} \vdash U \sqcap V$ *if* $\mathcal{R} \vdash U$ *and* $\mathcal{R} \vdash V$, *and* $\mathcal{R} \nvdash U \sqcap V$ *otherwise,*
- $\mathcal{R} \vdash U \sqcup V$ *if* $\mathcal{R} \vdash U$ *or* $\mathcal{R} \vdash V$, *and* $\mathcal{R} \nvdash U \sqcup V$ *otherwise.*

A Boolean role expression U *is* restricted *if* $\emptyset \nvdash U$. *The set of all restricted role expressions is denoted* $\mathbf{T}$, *and the symbols* U *and* V *will be used throughout this paper to denote restricted role expressions. A* $\mathcal{SHIQ}b$ Rbox *is a set of axioms of the form* $U \sqsubseteq V$ *(role inclusion axiom) or* $\mathsf{Tra}(R)$ *(transitivity axiom). The set of non-simple roles (for a given Rbox) is inductively defined as follows:*

- *If there is an axiom* $\mathsf{Tra}(R)$, *then* R *is non-simple.*
- *If there is an axiom* $R \sqsubseteq S$ *with* R *non-simple, then* S *is non-simple.*
- *If* R *is non-simple, then* $\mathrm{Inv}(R)$ *is non-simple.*

A role is simple *if it is atomic and not non-simple.*[6]
Based on a $\mathcal{SHIQ}b$ *Rbox, the set of* concept expressions $\mathbf{C}$ *is defined as follows:*

- $\mathsf{N}_C \subseteq \mathbf{C}$, $\top \in \mathbf{C}$, $\bot \in \mathbf{C}$,
- *if* $C, D \in \mathbf{C}$, $U \in \mathbf{T}$, $R \in \mathbf{R}$ *a simple role, and* n *a non-negative integer, then* $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall U.C$, $\exists U.C$, $\leq n\, R.C$, *and* $\geq n\, R.C$ *are also concept expressions.*

Throughout this paper, the symbols C, D *will be used to denote concept expressions. A* $\mathcal{SHIQ}b$ Tbox *is a set of* general concept inclusion axioms *(GCIs) of the form* $C \sqsubseteq D$.
A $\mathcal{SHIQ}b$ knowledge base KB *is the union of a* $\mathcal{SHIQ}b$ *Rbox and an according* $\mathcal{SHIQ}b$ *Tbox.*

As mentioned above, we will consider only fragments of $\mathcal{SHIQ}b$. In particular, a $\mathcal{SHIQ}$ knowledge base is a $\mathcal{SHIQ}b$ knowledge base without Boolean role expressions, and an $\mathcal{ALCI}b$ knowledge base is a $\mathcal{SHIQ}b$ knowledge base that contains no Rbox axioms and no number restrictions (i.e. axioms of the form $\leq n\, R.C$ or $\geq n\, R.C$). The DL $\mathcal{ALCI}b$ has first been described by Tobies [10].

[6] We will not consider DLs with transitivity and Boolean role expressions, so questioning the simplicity of such expressions is not relevant here.

Table 1. Semantics of concept constructors in $\mathcal{SHIQ}b$ for an interpretation $\mathcal{I}$ with domain $\Delta^\mathcal{I}$.

Name	Syntax	Semantics
inverse role	R^-	$\{\langle x,y\rangle \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid \langle y,x\rangle \in R^\mathcal{I}\}$
role negation	$\neg U$	$\{\langle x,y\rangle \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid \langle x,y\rangle \notin U^\mathcal{I}\}$
role conjunction	$U \sqcap V$	$U^\mathcal{I} \cap V^\mathcal{I}$
role disjunction	$U \sqcup V$	$U^\mathcal{I} \cup V^\mathcal{I}$
top	$\top$	$\Delta^\mathcal{I}$
bottom	$\bot$	$\emptyset$
negation	$\neg C$	$\Delta^\mathcal{I} \setminus C^\mathcal{I}$
conjunction	$C \sqcap D$	$C^\mathcal{I} \cap D^\mathcal{I}$
disjunction	$C \sqcup D$	$C^\mathcal{I} \cup D^\mathcal{I}$
univ. restriction	$\forall U.C$	$\{x \in \Delta^\mathcal{I} \mid \langle x,y\rangle \in U^\mathcal{I}$ implies $y \in C^\mathcal{I}\}$
exist. restriction	$\exists U.C$	$\{x \in \Delta^\mathcal{I} \mid$ for some $y \in \Delta^\mathcal{I}$, $\langle x,y\rangle \in U^\mathcal{I}$ and $y \in C^\mathcal{I}\}$
qualified number	$\leq n\, R.C$	$\{x \in \Delta^\mathcal{I} \mid \#\{y \in \Delta^\mathcal{I} \mid \langle x,y\rangle \in R^\mathcal{I}$ and $y \in C^\mathcal{I}\} \leq n\}$
restriction	$\geq n\, R.C$	$\{x \in \Delta^\mathcal{I} \mid \#\{y \in \Delta^\mathcal{I} \mid \langle x,y\rangle \in R^\mathcal{I}$ and $y \in C^\mathcal{I}\} \geq n\}$

Definition 2. *An interpretation $\mathcal{I}$ consists of a set $\Delta^\mathcal{I}$ called* domain *(the elements of it being called* individuals*) together with a function $\cdot^\mathcal{I}$ mapping*

- *individual names to elements of $\Delta^\mathcal{I}$,*
- *concept names to subsets of $\Delta^\mathcal{I}$, and*
- *role names to subsets of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$.*

The function $\cdot^\mathcal{I}$ is inductively extended to role and concept expressions as shown in Table 1. An interpretation $\mathcal{I}$ satisfies *an axiom φ if we find that $\mathcal{I} \models \varphi$:*

- *$\mathcal{I} \models U \sqsubseteq V$ if $U^\mathcal{I} \subseteq V^\mathcal{I}$,*
- *$\mathcal{I} \models \mathsf{Tra}(R)$ if $R^\mathcal{I}$ is a transitive relation,*
- *$\mathcal{I} \models C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$.*

An interpretation $\mathcal{I}$ satisfies *a knowledge base* KB *(we then also say that $\mathcal{I}$ is a* model *of* KB *and write $\mathcal{I} \models$* KB*) if it satisfies all axioms of* KB*. A knowledge base* KB *is* satisfiable *if it has a model. Two knowledge bases are* equivalent *if they have exactly the same models, and they are* equisatisfiable *if either both are unsatisfiable or both are satisfiable.*

For convenience of notation, we abbreviate Tbox axioms of the form $\top \sqsubseteq C$ by writing just C. Statements such as $\mathcal{I} \models C$ and $C \in$ KB are interpreted accordingly. Note that arbitrary GCIs $C \sqsubseteq D$ can thus be written as $\neg C \sqcup D$.
Finally, we will often need to access a particular set of quantified and atomic subformulae of a DL concept. These specific parts are provided by the function $P : \mathbf{C} \to 2^{\mathbf{C}}$:

$$P(C) := \begin{cases} P(D) & \text{if } C = \neg D \\ P(D) \cup P(E) & \text{if } C = D \sqcap E \text{ or } C = D \sqcup E \\ \{C\} \cup P(D) & \text{if } C = \text{Q}U.D \text{ with } \text{Q} \in \{\exists, \forall, \geq n, \leq n\} \\ \{C\} & \text{otherwise} \end{cases}$$

We generalise P to DL knowledge bases KB by defining $P(\text{KB})$ to be the union of the sets $P(C)$ for all Tbox axioms C of KB.

2.2 Knowledge Base Transformations

For our further considerations, we will usually express all Tbox axioms as single concept expressions as explained above. Given a knowledge base KB we obtain its negation normal form $\mathsf{NNF}(\mathrm{KB})$ by converting every Tbox concept into its negation normal form as usual:

$$
\begin{array}{lcl}
\mathsf{NNF}(\neg\top) & := & \bot \\
\mathsf{NNF}(\neg\bot) & := & \top \\
\mathsf{NNF}(C) & := & C \text{ if } C \in \{A, \neg A, \top, \bot\} \\
\mathsf{NNF}(\neg\neg C) & := & \mathsf{NNF}(C) \\
\mathsf{NNF}(C \sqcap D) & := & \mathsf{NNF}(C) \sqcap \mathsf{NNF}(D) \\
\mathsf{NNF}(\neg(C \sqcap D)) & := & \mathsf{NNF}(\neg C) \sqcup \mathsf{NNF}(\neg D) \\
\mathsf{NNF}(C \sqcup D) & := & \mathsf{NNF}(C) \sqcup \mathsf{NNF}(D) \\
\mathsf{NNF}(\neg(C \sqcup D)) & := & \mathsf{NNF}(\neg C) \sqcap \mathsf{NNF}(\neg D) \\
\mathsf{NNF}(\forall U.C) & := & \forall U.\mathsf{NNF}(C) \\
\mathsf{NNF}(\neg\forall U.C) & := & \exists U.\mathsf{NNF}(\neg C) \\
\mathsf{NNF}(\exists U.C) & := & \exists U.\mathsf{NNF}(C) \\
\mathsf{NNF}(\neg\exists U.C) & := & \forall U.\mathsf{NNF}(\neg C) \\
\mathsf{NNF}(\leq n\, R.C) & := & \leq n\, R.\mathsf{NNF}(C) \\
\mathsf{NNF}(\neg \leq n\, R.C) & := & \geq (n+1)\, R.\mathsf{NNF}(C) \\
\mathsf{NNF}(\geq n\, R.C) & := & \geq n\, R.\mathsf{NNF}(C) \\
\mathsf{NNF}(\neg \geq n\, R.C) & := & \leq (n-1)\, R.\mathsf{NNF}(C)
\end{array}
$$

It is well known that KB and $\mathsf{NNF}(\mathrm{KB})$ are equivalent. We will usually require another normalisation step that simplifies the structure of KB by *flattening* it to a knowledge base $\mathsf{FLAT}(\mathrm{KB})$. This is achieved by transforming KB into negation normal form and exhaustively applying the following transformation rules:

- Select an outermost occurrence of $\mathsf{Q}U.D$ in KB, such that $\mathsf{Q} \in \{\exists, \forall, \geq n, \leq n\}$ and D is a non-atomic concept.
- Substitute this occurrence with $\mathsf{Q}U.F$ where F is a fresh concept name (i.e. one not occurring in the knowledge base).
- If $\mathsf{Q} \in \{\exists, \forall, \geq n\}$, add $\neg F \sqcup D$ to the knowledge base.
- If $\mathsf{Q} = \leq n$ add $\mathsf{NNF}(\neg D) \sqcup F$ to the knowledge base.

Obviously, this procedure terminates yielding flat knowledge base $\mathsf{FLAT}(\mathrm{KB})$ all Tbox axioms of which are Boolean expressions over formulae of the form $\top$, $\bot$, A, $\neg A$, or $\mathsf{Q}U.A$ with A an atomic concept name.

Proposition 1. *Any* $\mathcal{SHIQ}b$ *knowledge base* KB *is equisatisfiable to* $\mathsf{FLAT}(\mathrm{KB})$.

Proof. We first prove inductively that every model of $\mathsf{FLAT}(\mathrm{KB})$ is a model of KB. Let KB′ be an intermediate knowledge base and let KB″ be the result of applying one single substitution step to KB′ as described in the above procedure. We now show that any model $\mathcal{I}$ of KB″ is a model of KB′. Let $\mathsf{Q}U.D$ be the term substituted in KB′. Note that after every substitution step, the knowledge base is still in negation normal form. Thus, we see that $\mathsf{Q}U.D$ occurs outside the scope of any negation or quantifier in an

KB′-axiom E', the same is the case for $\mathsf{Q}U.F$ in the respective KB″-axiom E'' obtained after the substitution. Hence, if we show that $(\mathsf{Q}U.F)^{\mathcal{I}} \subseteq (\mathsf{Q}U.D)^{\mathcal{I}}$, we can conclude that $E''^{\mathcal{I}} \subseteq E'^{\mathcal{I}}$. From $\mathcal{I}$ being a model of KB″ and therefore $E''^{\mathcal{I}} = \Delta^{\mathcal{I}}$, we would then easily derive that $E'^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and hence find that $\mathcal{I} \models \mathrm{KB}'$, as all other axioms from KB′ are trivially satisfied due to their presence in KB″.
It remains to show $(\mathsf{Q}U.F)^{\mathcal{I}} \subseteq (\mathsf{Q}U.D)^{\mathcal{I}}$. We distinguish four cases:

- $\mathsf{Q} = \exists$
  Consider a $\delta \in (\exists U.F)^{\mathcal{I}}$. Then exists an individual $\delta' \in \Delta^{\mathcal{I}}$ with $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$ and $\delta' \in F^{\mathcal{I}}$. As a consequence of the KB″ axiom $\neg F \sqcup D$ (being equivalent to the GCI $F \sqsubseteq D$), we find that $\delta' \in D^{\mathcal{I}}$ as well, leading straightforwardly to the conclusion $\delta \in (\exists U.D)^{\mathcal{I}}$. Hence we have $(\exists U.F)^{\mathcal{I}} \subseteq (\exists U.D)^{\mathcal{I}}$.
- $\mathsf{Q} = \forall$
  Consider a $\delta \in (\forall U.F)^{\mathcal{I}}$. This implies for every individual $\delta' \in \Delta^{\mathcal{I}}$ with $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$ that $\delta' \in F^{\mathcal{I}}$. Again, the KB″ axiom $\neg F \sqcup D$ entails $\delta' \in D^{\mathcal{I}}$ for every such δ', leading to $\delta \in (\forall U.D)^{\mathcal{I}}$. Hence, we have $(\forall U.F)^{\mathcal{I}} \subseteq (\forall U.D)^{\mathcal{I}}$.
- $\mathsf{Q} = \geq n$
  Consider a $\delta \in (\geq n\, U.F)^{\mathcal{I}}$. This means there are distinct individuals $\delta_1, \ldots, \delta_n \in \Delta^{\mathcal{I}}$ with $\langle \delta, \delta_i \rangle \in U^{\mathcal{I}}$ and $\delta_i \in F^{\mathcal{I}}$ for $1 \leq i \leq n$. As a consequence of the KB″ axiom $\neg F \sqcup D$, we find that $\delta_i \in D^{\mathcal{I}}$ for all the n distinct δ_i, and conclude $\delta \in (\geq n\, U.F)^{\mathcal{I}}$. Hence, we have $(\geq n\, U.F)^{\mathcal{I}} \subseteq (\geq n\, U.D)^{\mathcal{I}}$.
- $\mathsf{Q} = \leq n$
  Consider a $\delta \in (\leq n\, U.F)^{\mathcal{I}}$. This implies that the number of individuals $\delta' \in \Delta^{\mathcal{I}}$ with $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$ and $\delta' \in F^{\mathcal{I}}$ is not greater than n. By the KB″ axiom $\mathsf{NNF}(\neg D) \sqcup F$ (being equivalent to the GCI $D \sqsubseteq F$), we know $D^{\mathcal{I}} \subseteq F^{\mathcal{I}}$. Thus, also the number of individuals $\delta' \in \Delta^{\mathcal{I}}$ with $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$ and $\delta' \in D^{\mathcal{I}}$ cannot be greater than n, leading to the conclusion $\delta \in (\leq n\, U.D)^{\mathcal{I}}$. Hence, we have $(\leq n\, U.F)^{\mathcal{I}} \subseteq (\leq n\, U.D)^{\mathcal{I}}$.

Every model $\mathcal{I}$ of KB can be transformed into a model $\mathcal{J}$ of FLAT(KB) by following the flattening process described above: Let KB″ result from KB′ by substituting $\mathsf{Q}U.D$ by $\mathsf{Q}U.F$ and adding the respective axiom. Furthermore, let $\mathcal{I}'$ be a model of KB′. Now we construct the interpretation $\mathcal{I}''$ as follows: $F^{\mathcal{I}''} := (\mathsf{Q}U.D)^{\mathcal{I}'}$ and for all other concept and role names N we set $N^{\mathcal{I}''} := N^{\mathcal{I}'}$. Then $\mathcal{I}''$ is a model of KB″. □

3 Building Models from Domino Sets

In this section, we introduce the notion of a set of *dominoes* for a given terminological $\mathcal{ALCI}b$ knowledge base. Intuitively, each domino abstractly represents two individuals in an $\mathcal{ALCI}b$ interpretation, based on their concept properties and mutual role relationships. We will see that suitable sets of such two-element pieces suffice to reconstruct models of $\mathcal{ALCI}b$, which also reveals certain model theoretic properties of this not so common DL. In particular, every satisfiable $\mathcal{ALCI}b$ Tbox admits tree-shaped models. This result is rather a by-product of our main goal of decomposing models into unstructured sets of local domino components, but it explains why our below constructions have some similarity with common approaches of showing tree-model properties by “unravelling” models.

After introducing the basics of domino representation, we present an algorithm for deciding satisfiability of an $\mathcal{ALCIb}$ terminology based on sets of dominoes.

3.1 From Interpretations to Dominoes

We now introduce the basic notion of a domino set, and its relationship to interpretations. Given a DL language with concepts **C** and roles **R**, a *domino* is an arbitrary triple $\langle\mathcal{A},\mathcal{R},\mathcal{B}\rangle$, where $\mathcal{A},\mathcal{B}\subseteq\mathbf{C}$ and $\mathcal{R}\subseteq\mathbf{R}$. In the following, we will always assume a fixed language and refer to dominoes over that language only.
We now formalise the idea of deconstructing an interpretation into a set of dominoes.

Definition 3. *Given an interpretation $\mathcal{I}=\langle\Delta^{\mathcal{I}},\cdot^{\mathcal{I}}\rangle$, and a set $C\subseteq\mathbf{C}$ of concept expressions, the* domino projection *of $\mathcal{I}$ w.r.t. C, denoted by $\pi_C(\mathcal{I})$ is the set that contains for all $\delta,\delta'\in\Delta^{\mathcal{I}}$ the triple $\langle\mathcal{A},\mathcal{R},\mathcal{B}\rangle$ with*

- $\mathcal{A}=\{C\in C\mid\delta\in C^{\mathcal{I}}\}$,
- $\mathcal{R}=\{R\in\mathbf{R}\mid\langle\delta,\delta'\rangle\in R^{\mathcal{I}}\}$
- $\mathcal{B}=\{C\in C\mid\delta'\in C^{\mathcal{I}}\}$.

It is easy to see that domino projections do not faithfully represent the structure of the interpretation that they were constructed from. But as we will see below, domino projections capture enough information to reconstruct models of a knowledge base KB, as long as C is chosen to contain at least $P(\mathrm{KB})$. For this purpose, we now introduce the inverse construction of interpretations from arbitrary domino sets.

Definition 4. *Given a set $\mathbb{D}$ of dominoes, the induced* domino interpretation $\mathcal{I}(\mathbb{D})=\langle\Delta^{\mathcal{I}},\cdot^{\mathcal{I}}\rangle$ *is defined as follows:*

1. *$\Delta^{\mathcal{I}}$ consists of all nonempty finite words over $\mathbb{D}$ where, for each pair of subsequent letters $\langle\mathcal{A},\mathcal{R},\mathcal{B}\rangle$ and $\langle\mathcal{A}',\mathcal{R}',\mathcal{B}'\rangle$ in a word, we have $\mathcal{B}=\mathcal{A}'$.*
2. *For a word $\delta=\langle\mathcal{A}_1,\mathcal{R}_1,\mathcal{A}_2\rangle\langle\mathcal{A}_2,\mathcal{R}_2,\mathcal{A}_3\rangle\ldots\langle\mathcal{A}_{i-1},\mathcal{R}_{i-1},\mathcal{A}_i\rangle$ and a concept name $A\in\mathsf{N}_C$, we define $\mathrm{tail}(\delta):=\mathcal{A}_i$, and set $\delta\in A^{\mathcal{I}}$ iff $A\in\mathrm{tail}(\delta)$,*
3. *For a role name $R\in\mathsf{N}_R$, we set $\langle\delta_1,\delta_2\rangle\in R^{\mathcal{I}}$ if either*

$$\delta_2=\delta_1\langle\mathcal{A},\mathcal{R},\mathcal{B}\rangle \text{ with } R\in\mathcal{R} \quad\text{or}\quad \delta_1=\delta_2\langle\mathcal{A},\mathcal{R},\mathcal{B}\rangle \text{ with } \mathrm{Inv}(R)\in\mathcal{R}.$$

We are now ready to show that certain domino projections contain enough information to reconstruct models of a knowledge base.

Proposition 2. *Consider a set $C\subseteq\mathbf{C}$ of concept expressions, and an interpretation $\mathcal{J}$, and let $\mathcal{K}:=\mathcal{I}(\pi_C(\mathcal{J}))$ denote the interpretation of the domino projection of $\mathcal{J}$ w.r.t. C. Then, for any $\mathcal{ALCIb}$ concept expression $C\in\mathbf{C}$ with $P(C)\subseteq C$, we have that $\mathcal{J}\models C$ iff $\mathcal{K}\models C$.*
Especially, for any $\mathcal{ALCIb}$ knowledge base KB, $\mathcal{J}\models\mathrm{KB}$ iff $\mathcal{I}(\pi_{P(\mathrm{KB})}(\mathcal{J}))\models\mathrm{KB}$.

Proof. We first show the following: Given any $\mathcal{J}$-individual δ and $\mathcal{K}$-individual ϵ such that $\mathrm{tail}(\epsilon)=\{D\in C\mid\delta\in D^{\mathcal{J}}\}$, we find that $\epsilon\in C^{\mathcal{K}}$ iff $\delta\in C^{\mathcal{J}}$. Clearly, the overall claim follows from that statement using the observation that a suitable $\delta\in\Delta^{\mathcal{K}}$ must

exist for all $\epsilon \in \Delta^{\mathcal{K}}$ and vice versa. We proceed by induction over the structure of C, noting that $P(C) \subseteq \mathcal{C}$ implies $P(D) \subseteq \mathcal{C}$ for any subconcept D of C.
The base case $C \in \mathsf{N}_C$ is immediately satisfied by our assumption on the relationship of δ and ϵ. For the induction step, we first note that the case $C \in \{\top, \bot\}$ is also trivial. For $C = \neg D$ and $C = D \sqcap D'$ as well as $C = D \sqcup D'$, the claim follows immediately from the induction hypothesis for D and D'.
Next consider the case $C = \exists U.D$, and assume that $\delta \in C^{\mathcal{J}}$. Hence there is some $\delta' \in \Delta^{\mathcal{J}}$ such that $\langle \delta, \delta' \rangle \in U^{\mathcal{J}}$ and $\delta' \in D^{\mathcal{J}}$. Then the pair $\langle \delta, \delta' \rangle$ generates a domino $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ and $\Delta^{\mathcal{K}}$ constains $\epsilon' = \epsilon \langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$. $\langle \delta, \delta' \rangle \in U^{\mathcal{J}}$ implies $\mathcal{R} \vdash U$, and hence $\langle \epsilon, \epsilon' \rangle \in U^{\mathcal{K}}$. Applying the induction hypothesis to D, we conclude $\epsilon' \in D^{\mathcal{K}}$. Now $\epsilon \in C^{\mathcal{K}}$ follows from the construction of $\mathcal{K}$.
For the converse, assume that $\epsilon \in C^{\mathcal{K}}$. Hence there is some $\epsilon' \in \Delta^{\mathcal{K}}$ such that $\langle \epsilon, \epsilon' \rangle \in U^{\mathcal{K}}$ and $\epsilon' \in D^{\mathcal{K}}$. By the definition of $\mathcal{K}$, there are two possible cases:

- $\epsilon' = \epsilon \langle \mathrm{tail}(\epsilon), \mathcal{R}, \mathrm{tail}(\epsilon') \rangle$ and $\mathcal{R} \vdash U$: Consider the two $\mathcal{J}$-individuals $\langle \delta', \delta'' \rangle$ generating the domino $\langle \mathrm{tail}(\epsilon), \mathcal{R}, \mathrm{tail}(\epsilon') \rangle$. From $\epsilon' \in D^{\mathcal{K}}$ and the induction hypothesis, we obtain $\delta'' \in D^{\mathcal{J}}$. Together with $\langle \delta', \delta'' \rangle \in U^{\mathcal{J}}$ this implies $\delta' \in C^{\mathcal{J}}$. Since $C = \exists U.D \in \mathcal{C}$, we also have $C \in \mathrm{tail}(\epsilon)$ and thus $\delta \in C^{\mathcal{J}}$ as claimed.
- $\epsilon = \epsilon' \langle \mathrm{tail}(\epsilon'), \mathcal{R}, \mathrm{tail}(\epsilon) \rangle$ and $\mathrm{Inv}(\mathcal{R}) \vdash U$: This case is similar to the first case, merely exchanging the order of $\langle \delta', \delta'' \rangle$ and using $\mathrm{Inv}(\mathcal{R})$ instead of $\mathcal{R}$.

Finally, the case $C = \forall U.D$ is dual to the case $C = \exists U.D$, and we will omit the repeated argument. Note, however, that this case does not follow from the semantic equivalence of $\forall U.D$ and $\neg \exists U.\neg D$, since the proof hinges upon the inclusion of $\neg D$ in $\mathcal{C}$ which is not given directly. □

3.2 Constructing Domino Sets

As shown in the previous section, the domino projection of a model of an $\mathcal{ALCI}b$ knowledge base can contain enough information to allow for the reconstruction of a model. This observation can be the basis for designing an algorithm that decides knowledge base satisfiability. Usually (especially in tableau-based algorithms), checking satisfiability amounts to the attempt to construct a (representation of a) model. As we have seen, in our case it suffices to try to construct just a model's domino projection. If this can be done, we know that there is a model, if not, there is none.
In what follows, we first describe the iterative construction of such a domino set from a given knowledge base, and then show that it is indeed a decision procedure for knowledge base satisfiability.

Definition 5. *Consider an $\mathcal{ALCI}b$ knowledge base* KB*, and define $\mathcal{C} = P(\mathsf{FLAT}(\mathrm{KB}))$. Sets $\mathbb{D}_i$ of dominoes based on concepts from $\mathcal{C}$ are constructed as follows:*
$\mathbb{D}_0$ consists of all dominoes $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ satisfying the following conditions:

kb: *for every concept $C \in \mathsf{FLAT}(\mathrm{KB})$, we have that $\bigsqcap_{D \in \mathcal{A}} D \sqsubseteq C$ is a tautology*[7]*,*

[7] Please note that the formulae in $\mathsf{FLAT}(\mathrm{KB})$ and in $\mathcal{A} \subseteq \mathcal{C}$ are such that this can easily be checked by evaluating the Boolean operators in C as if $\mathcal{A}$ was a set of true propositional variables.

ex: *for all* $\exists U.A \in C$ *with* $A \in \mathcal{B}$ *and* $\mathcal{R} \vdash U$, *we have* $\exists U.A \in \mathcal{A}$,
uni: *for all* $\forall U.A \in C$ *with* $\forall U.A \in \mathcal{A}$ *and* $\mathcal{R} \vdash U$ *we have* $A \in \mathcal{B}$.

Given a domino set $\mathbb{D}_i$, *the set* $\mathbb{D}_{i+1}$ *consists of all dominoes* $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle \in \mathbb{D}_i$ *satisfying the following conditions:*

delex: *for every* $\exists U.A \in C$ *with* $\exists U.A \in \mathcal{A}$, *there is some* $\langle \mathcal{A}, \mathcal{R}', \mathcal{B}' \rangle \in \mathbb{D}_i$ *such that* $\mathcal{R}' \vdash U$ *and* $A \in \mathcal{B}'$,
deluni: *for every* $\forall U.A \in C$ *with* $\forall U.A \notin \mathcal{A}$, *there is some* $\langle \mathcal{A}, \mathcal{R}', \mathcal{B}' \rangle \in \mathbb{D}_i$ *such that* $\mathcal{R}' \vdash U$ *but* $A \notin \mathcal{B}'$,
sym: $\langle \mathcal{B}, \mathrm{Inv}(\mathcal{R}), \mathcal{A} \rangle \in \mathbb{D}_i$.

The construction of domino sets $\mathbb{D}_{i+1}$ *is continued until* $\mathbb{D}_{i+1} = \mathbb{D}_i$. *The final result* $\mathbb{D}_{\mathrm{KB}} := \mathbb{D}_{i+1}$ *defines the* canonical domino set *of* KB.
The algorithm returns "unsatisfiable" if $\mathbb{D}_{\mathrm{KB}} = \emptyset$, *and "satisfiable" otherwise.*

Note that $\mathbb{D}_0$ is exponential in the size of the knowledge base, such that the iterative deletion of dominoes must terminate after at most exponentially many steps. Below we will show that this procedure is indeed sound and complete for checking satisfiability.
Note that, in contrast to tableau procedures, the presented algorithm starts with a large set of dominoes and successively deletes undesired dominoes. Indeed, we will soon show that the constructed domino set is the largest such set from which a domino model can be obtained. The algorithm thus may seem to be of little practical use. In Section 4, we therefore refine the above algorithm to employ Boolean functions as efficient implicit representations of domino sets, such that the efficient computational methods of OBDDs can be exploited. In the meantime, however, domino sets will serve us well for showing the required correctness properties.
An important property of domino interpretations constructed from canonical domino sets is that the (semantic) concept membership of an individual can typically be (syntactically) read from the domino it has been constructed of.

Lemma 1. *Consider an* $\mathcal{ALCIb}$ *knowledge base* KB *with non-empty canonical domino set* $\mathbb{D}_{\mathrm{KB}}$, *and define* $C := P(\mathsf{FLAT}(\mathrm{KB}))$ *and* $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle := \mathcal{I}(\mathbb{D}_{\mathrm{KB}})$. *Then for all* $C \in C$ *and* $\delta \in \Delta^{\mathcal{I}}$, *we have that* $\delta \in C^{\mathcal{I}}$ *iff* $C \in \mathrm{tail}(\delta)$. *Moreover,* $\mathcal{I} \models \mathsf{FLAT}(\mathrm{KB})$.

Proof. First note that the domain of $\mathcal{I}$ is obviously non-empty whenever $\mathbb{D}_{\mathrm{KB}}$ is. Now if $C \in \mathsf{N}_C$ is an atomic concept, the first claim follows directly from the definition of $\mathcal{I}$. The remaining cases that may occur in $P(\mathsf{FLAT}(\mathrm{KB}))$ are $C = \exists U.A$ and $C = \forall U.A$.
First consider the case $C = \exists U.A$, and assume that $\delta \in C^{\mathcal{I}}$. Thus there is $\delta' \in \Delta^{\mathcal{I}}$ with $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$ and $\delta' \in A^{\mathcal{I}}$. The construction of the domino model admits two possible cases:

- $\delta' = \delta\langle \mathrm{tail}(\delta), \mathcal{R}, \mathrm{tail}(\delta') \rangle$ with $\mathcal{R} \vdash U$ and $A \in \mathrm{tail}(\delta')$. Since $\mathbb{D}_{\mathrm{KB}} \subseteq \mathbb{D}_0$, we find that $\langle \mathrm{tail}(\delta), \mathcal{R}, \mathrm{tail}(\delta') \rangle$ satisfies condition **ex**, and thus $C \in \mathrm{tail}(\delta)$ as required.
- $\delta = \delta'\langle \mathrm{tail}(\delta'), \mathcal{R}, \mathrm{tail}(\delta) \rangle$ with $\mathrm{Inv}(\mathcal{R}) \vdash U$ and $A \in \mathrm{tail}\delta'$. By condition **sym**, $\mathbb{D}_{\mathrm{KB}}$ also contains the domino $\langle \mathrm{tail}(\delta), \mathrm{Inv}(\mathcal{R}), \mathrm{tail}(\delta') \rangle$, and we can again invoke **ex** to conclude $C \in \mathrm{tail}(\delta)$.

For the other direction, assume that $\exists U.A \in \text{tail}(\delta)$. Thus $\mathbb{D}_{\text{KB}}$ contains some domino $\langle \mathcal{A}, \mathcal{R}, \text{tail}(\delta) \rangle$, and by **sym** also the domino $\langle \text{tail}(\delta), \mathcal{R}, \mathcal{A} \rangle$. By condition **delex**, the latter implies that $\mathbb{D}_{\text{KB}}$ contains a domino $\langle \text{tail}(\delta), \mathcal{R}', \mathcal{A}' \rangle$. According to **delex**, we find that $\delta' = \delta \langle \text{tail}(\delta), \mathcal{R}', \mathcal{A}' \rangle$ is an $\mathcal{I}$-individual such that $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$ and $\delta' \in A^{\mathcal{I}}$. Thus $\delta \in (\exists U.A)^{\mathcal{I}}$ as claimed.

For the second case, consider $C = \forall U.A$ and assume that $\delta \in C^{\mathcal{I}}$, and thus $\mathbb{D}_{\text{KB}}$ contains some domino $\langle \mathcal{A}, \mathcal{R}, \text{tail}(\delta) \rangle$, and by **sym** also the domino $\langle \text{tail}(\delta), \mathcal{R}, \mathcal{A} \rangle$. For a contradiction, suppose that $\forall U.A \notin \text{tail}(\delta)$. By condition **deluni**, the latter implies that $\mathbb{D}_{\text{KB}}$ contains a domino $\langle \text{tail}(\delta), \mathcal{R}', \mathcal{A}' \rangle$. According to **deluni**, we find that $\delta' = \delta \langle \text{tail}(\delta), \mathcal{R}', \mathcal{A}' \rangle$ is an $\mathcal{I}$-individual such that $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$ and $\delta' \notin D^{\mathcal{I}}$. But then $\delta \notin (\forall U.A)^{\mathcal{I}}$, which is the required contradiction.

For the other direction, assume that $\forall U.A \in \text{tail}(\delta)$. According to the construction of the domino model, there are two possible cases for elements δ' with $\langle \delta, \delta' \rangle \in U^{\mathcal{I}}$:

- $\delta' = \delta \langle \text{tail}(\delta), \mathcal{R}, \text{tail}(\delta') \rangle$ with $\mathcal{R} \vdash U$. Since $\mathbb{D}_{\text{KB}} \subseteq \mathbb{D}_0$, $\langle \text{tail}(\delta), \mathcal{R}, \text{tail}(\delta') \rangle$ must satisfy condition **uni**, and thus $A \in \text{tail}(\delta')$.
- $\delta = \delta' \langle \text{tail}(\delta'), \mathcal{R}, \text{tail}(\delta) \rangle$ with $\text{Inv}(\mathcal{R}) \vdash U$. By condition **sym**, $\mathbb{D}_{\text{KB}}$ also contains the domino $\langle \text{tail}(\delta), \text{Inv}(\mathcal{R}), \text{tail}(\delta') \rangle$, and we can again invoke **uni** to conclude $A \in \text{tail}(\delta')$.

Thus, $A \in \text{tail}(\delta')$ for all U-successors δ' of δ, and hence $\delta \in (\forall U.A)^{\mathcal{I}}$ as claimed.

For the rest of the claim, note that any domino $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ must satisfy condition **kb**. Using condition **sym**, we conclude that for any $\delta \in \Delta^{\mathcal{I}}$, the axiom $\bigsqcap_{D \in \text{tail}(\delta)} D \sqsubseteq C$ is a tautology for all $C \in \mathsf{FLAT}(\text{KB})$. As shown above, $\delta \in D^{\mathcal{I}}$ for all $D \in \text{tail}(\delta)$, and thus $\delta \in C$. Hence every individual of $\mathcal{I}$ is an instance of each concept of $\mathsf{FLAT}(\text{KB})$ as required. □

The previous lemma shows soundness of our decision algorithm. Conversely, completeness is shown by the following lemma.

Lemma 2. *Consider an $\mathcal{ALCIb}$ knowledge base* KB*. If* KB *is satisfiable, then its canonical domino set $\mathbb{D}_{\text{KB}}$ is non-empty.*

Proof. Consider any model $\mathcal{I}$ of KB. A simple induction shows that the domino projection $\pi_{P(\mathsf{FLAT}(\text{KB}))}(\mathcal{I})$ is contained in $\mathbb{D}_{\text{KB}}$. In the following, we use $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ to denote an arbitrary domino of $\pi_{P(\mathsf{FLAT}(\text{KB}))}(\mathcal{I})$.

For the base case, we must show that $\pi_{P(\mathsf{FLAT}(\text{KB}))}(\mathcal{I}) \subseteq \mathbb{D}_0$. Let $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ to denote an arbitrary domino of $\pi_{P(\mathsf{FLAT}(\text{KB}))}(\mathcal{I})$ which was generated from elements $\langle \delta, \delta' \rangle$. Then $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ satisfies condition **kb**, since $\delta \in C^{\mathcal{I}}$ for any $C \in \mathsf{FLAT}(\text{KB})$. The conditions **ex** and **uni** are obviously satisfied.

For the induction step, assume that $\pi_{P(\mathsf{FLAT}(\text{KB}))}(\mathcal{I}) \subseteq \mathbb{D}_i$, and let $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ again denote an arbitrary domino of $\pi_{P(\mathsf{FLAT}(\text{KB}))}(\mathcal{I})$ which was generated from elements $\langle \delta, \delta' \rangle$.

- For **delex**, note that $\exists U.A \in \mathcal{A}$ implies $\delta \in (\exists U.A)^{\mathcal{I}}$. Thus there is an individual δ'' such that $\langle \delta, \delta'' \rangle \in U^{\mathcal{I}}$ and $\delta'' \in A^{\mathcal{I}}$. Clearly, the domino generated by $\langle \delta, \delta'' \rangle$ satisfies the conditions of **delex**.

– For **deluni**, note that $\forall U.A \notin \mathcal{A}$ implies $\delta \notin (\forall U.A)^{\mathcal{I}}$. Thus there is an individual δ'' such that $\langle \delta, \delta'' \rangle \in U^{\mathcal{I}}$ and $\delta'' \notin A^{\mathcal{I}}$. Clearly, the domino generated by $\langle \delta, \delta'' \rangle$ satisfies the conditions of **deluni**.
– The condition of **sym** for $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle$ is clearly satisfied by the domino generated from $\langle \delta', \delta \rangle$.

□

Combining the results of Lemma 1, Proposition 1, and Lemma 2, we obtain the main result of this section:

Theorem 1. *A terminological $\mathcal{ALCI}b$ knowledge base* KB *is satisfiable iff its canonical domino set $\mathbb{D}_{\text{KB}}$ is non-empty. Definition 5 thus defines a decision procedure for satisfiability of such $\mathcal{ALCI}b$ knowledge bases.*

4 Sets as Boolean Functions

In this section, we explain how large sets (in our case the canonical domino, respectively the intermediate sets during its construction) can be effectively represented implicitly via Boolean functions. This kind of encoding is rather standard within the field of OBDD-based model checking, and so we will only give a very brief overview on OBDDs and not further elaborate on the technical details of their manipulation in this paper. The way of implementing our approach, however, can be directly derived from the algorithm described in this section, as for every operation to be carried out on the Boolean functions (namely combining them, permutation of variables, instantiating variables etc.) there is an algorithmic counterpart for the OBDD-based representation.

4.1 Boolean Functions and Operations

We will start with a brief introduction of how sets can be represented by means of Boolean functions. This will enable us, given a fixed finite base set S, to represent every family of sets $\mathbb{S} \subseteq 2^S$ by a single Boolean function.
A *Boolean function* on a set Var of variables is a function $\varphi : 2^{\mathsf{Var}} \to \{true, false\}$. The underlying intuition is that $\varphi(V)$ computes the truth value of a Boolean formula based on the assumption that exactly the variables of V are evaluated to *true*. A simple example are functions of the form $[\![v]\!]$ for some $v \in \mathsf{Var}$, which are defined by setting $[\![v]\!](V) := true$ iff $v \in V$.
Boolean functions over the same set of variables can be combined and modified in several ways. Firstly, there are the obvious Boolean operators for negation, conjunction, disjunction, and implication. By slight abuse of notation, we will use the common (syntactic) operator symbols $\neg$, $\wedge$, $\vee$, and $\to$ to also represent such (semantic) operators on Boolean functions. For example, given Boolean functions φ and ψ, we find that $(\varphi \wedge \psi)(V) = true$ iff $\varphi(V) = true$ and $\psi(V) = true$. Note that the result of the application of $\wedge$ results in another Boolean function, and is not to be understood as a syntactic formula.

Another operation on Boolean functions is existential quantification over a set of variables $V \subseteq \mathsf{Var}$, written as $\exists V.\varphi$ for some function φ. Given an input set $W \subseteq \mathsf{Var}$ of variables, we define $(\exists V.\varphi)(W) = \mathit{true}$ iff *there is some* set $V' \subseteq V$ such that $\varphi(V' \cup (W \setminus V)) = \mathit{true}$. In other words, there must be a way to set truth values of variables in V such that φ evaluates to *true*. Universal quantification is defined analogously, and we thus have $\forall V.\varphi := \neg\exists V.\neg\varphi$ as usual. Again, we remark that our use of $\exists$ and $\forall$ overloads notation, and should not be confused with role restrictions in DL expressions.

4.2 Ordered Binary Decision Diagrams

Ordered Binary Decision Diagrams are data structures that encode Boolean functions in an efficient way. Structurally, a *binary decision diagram* (BDD) is a directed acyclic graph with two distinguished nodes: the *true*-node and the *false*-node, also called terminal nodes. Moreover, one node without incoming edges is marked as root. All nodes except the terminal ones are labelled by a variable from the set Var and have exactly two outgoing edges, one of them labelled by *true* and the other by *false*. Every BDD based on a variable set $\mathsf{Var} = \{v_1, \ldots, v_n\}$ represents an n-ary boolean function $\varphi : 2^{\mathsf{Var}} \to \{\mathit{true}, \mathit{false}\}$ in the following way: for every variable subset $V \subseteq \mathsf{Var}$, the value $\varphi(V)$ is determined by a traversal of the BDD: starting from the root node, we take the variable v by which the actual node is labelled and follow the outgoing edge that is labelled by *true* if $v \in V$ and by *false* if $v \notin V$. This is iterated until a terminal node is reached, which then indicates the result of $\varphi(V)$. A BDD is called *ordered* BDD (short OBDD) if there is a total order on the set Var such that any path in the BDD is strictly ascending wrt. that order.

For any boolean function $\varphi : 2^{\mathsf{Var}} \to \{\mathit{true}, \mathit{false}\}$ and any ordering on Var there is (up to isomorphy) exactly one minimal OBDD realizing it. Moreover, this minimal representative (also called *reduced* OBDD, short ROBDD) can be efficiently computed from any non-minimal OBDD. This enables to efficiently decide whether two ROBDDs encode the same Boolean function. In particular, the ROBDD corresponding to the Boolean function assigning *false* to every input consists of just two nodes: the *false*-node, marked as root, and the (actually “unused”) *true*-node. This indicates that the data compression realiyed by OBDDs enables quick satisfiability tests.

While for a fixed order, the OBDD for a certain Boolean formula usually might get exponentially large, it is often possible to find an order where this is not the case. Finding the optimal order is NP-complete, however, heuristics have shown to yield good approximate solutions. Hence OBDDs can be conceived as a beneficial way to represent Boolean functions in a compressed way.

Moreover, also operations on Boolean functions (such as the aforementioned “pointwise” negation, conjunction, disjunction, implication as well as quantification over propositional variables) can be done directly on the corresponding OBDDs and there exist fast algorithms for doing so.

4.3 Translating Dominos into Boolean Functions

Now, let $\mathrm{KB} = \mathsf{FLAT}(\mathrm{KB})$ be a flattened $\mathcal{ALCIb}$ knowledge base. The variable set Var is defined as $\mathsf{Var} := \mathbf{R} \cup (P(\mathrm{KB}) \times \{1, 2\})$. We thus obtain an obvious bijection between sets $V \subseteq \mathsf{Var}$ and dominoes over the set $P(\mathrm{KB})$ given as $\langle \mathcal{A}, \mathcal{R}, \mathcal{B} \rangle \mapsto (\mathcal{A} \times \{1\}) \cup \mathcal{R} \cup (\mathcal{B} \times \{2\})$. Hence, any Boolean function over Var represents a domino set as the collection of all variable sets for which it evaluates to *true*. We can use this observation to rephrase the construction of $\mathbb{D}_{\mathrm{KB}}$ in Definition 5 into an equivalent construction of a function $[\![\mathrm{KB}]\!]$.

We represent DL concepts C and role expressions U by characteristic Boolean functions over Var as follows:

$$[\![C]\!] := \begin{cases} \neg[\![D]\!] & \text{if } C = \neg D \\ [\![D]\!] \wedge [\![E]\!] & \text{if } C = D \sqcap E \\ [\![D]\!] \vee [\![E]\!] & \text{if } C = D \sqcup E \\ [\![\langle C, 1\rangle]\!] & \text{if } C \in P(\mathrm{KB}) \end{cases} \qquad [\![U]\!] = \begin{cases} \neg[\![V]\!] & \text{if } U = \neg V \\ [\![V]\!] \wedge [\![W]\!] & \text{if } U = V \sqcap W \\ [\![V]\!] \vee [\![W]\!] & \text{if } U = V \sqcup W \\ [\![U]\!] & \text{if } U \in \mathbf{R} \end{cases}$$

We can now define an inferencing algorithm based on Boolean functions.

Definition 6. *Given a flattened* $\mathcal{ALCIb}$ *knowledge base* $\mathrm{KB} = \mathsf{FLAT}(\mathrm{KB})$ *and a variable set* Var *as above, iteratively construct Boolean functions* $[\![\mathrm{KB}]\!]_i$ *as follows:*

For $i = 0$*, initialise* $[\![\mathrm{KB}]\!]_0 := \varphi^{\mathbf{kb}} \wedge \varphi^{\mathbf{uni}} \wedge \varphi^{\mathbf{ex}}$*, where*

$$\varphi^{\mathbf{kb}} := \bigwedge_{C \in \mathrm{KB}} [\![C]\!]$$

$$\varphi^{\mathbf{uni}} := \bigwedge_{\forall U.C \in P(\mathrm{KB})} [\![\langle \forall U.C, 1\rangle]\!] \wedge [\![U]\!] \to [\![\langle C, 2\rangle]\!]$$

$$\varphi^{\mathbf{ex}} := \bigwedge_{\exists U.C \in P(\mathrm{KB})} [\![\langle C, 2\rangle]\!] \wedge [\![U]\!] \to [\![\langle \exists U.C, 1\rangle]\!]$$

For $i \geq 1$*, iteratively define* $[\![\mathrm{KB}]\!]_{i+1} := [\![\mathrm{KB}]\!]_i \wedge \varphi_i^{\mathbf{delex}} \wedge \varphi_i^{\mathbf{deluni}} \wedge \varphi_i^{\mathbf{sym}}$*, where*

$$\varphi_i^{\mathbf{delex}} := \bigwedge_{\exists U.C \in P(\mathrm{KB})} [\![\langle \exists U.C, 1\rangle]\!] \to \exists(\mathbf{R} \cup \mathcal{C} \times \{2\}).([\![\mathrm{KB}]\!]_i \wedge [\![U]\!] \wedge [\![\langle C, 2\rangle]\!])$$

$$\varphi_i^{\mathbf{deluni}} := \bigwedge_{\forall U.C \in P(\mathrm{KB})} [\![\langle \forall U.C, 1\rangle]\!] \to \neg\exists(\mathbf{R} \cup \mathcal{C} \times \{2\}).([\![\mathrm{KB}]\!]_i \wedge [\![U]\!] \wedge \neg[\![\langle C, 2\rangle]\!])$$

$$\varphi_i^{\mathbf{sym}}(V) := [\![\mathrm{KB}]\!]_i\Big(\{\langle D, 1\rangle \mid \langle D, 2\rangle \in V\} \cup \{\mathrm{Inv}(R) \mid R \in V\} \cup \{\langle D, 2\rangle \mid \langle D, 1\rangle \in V\}\Big)$$

After constructing a function $[\![\mathrm{KB}]\!]_{i+1}$*, check whether* $[\![\mathrm{KB}]\!]_{i+1} = [\![\mathrm{KB}]\!]_i$*. If this is the case, the result of the construction is defined as* $[\![\mathrm{KB}]\!] := [\![\mathrm{KB}]\!]_i$*. Otherwise, repeat the second construction step to obtain* $[\![\mathrm{KB}]\!]_{i+2}$.
After the construction has terminated, check whether $[\![\mathrm{KB}]\!] = [\![false]\!]$*, i.e. whether* $[\![\mathrm{KB}]\!](V) = false$ *for all* $V \subseteq \mathsf{Var}$*. If this is the case, return "unsatisfiable", otherwise return "satisfiable."*

The above algorithm is a correct procedure for checking consistency of terminological $\mathcal{ALCI}b$ knowledge bases – note that all necessary computation steps can indeed be implemented algorithmically: Any Boolean function can be evaluated for a fixed variable input V, and equality of two functions can (naively) be checked by comparing the results for all possible input sets (which are finitely many since Var is). Similarly, the algorithm terminates since the sequence is decreasing w.r.t $\{V \mid [\![\mathrm{KB}]\!]_i(V) = true\}$ and there can be only finitely many Boolean functions over Var.
Concerning soundness and completeness, it is easy to see that the Boolean operations used in constructing $[\![\mathrm{KB}]\!]$ directly correspond to the set operations in Definition 5, such that $[\![\mathrm{KB}]\!](V) = true$ iff V represents a domino in $\mathbb{D}_{\mathrm{KB}}$. Thus soundness and completeness is shown by Theorem 1.

5 Polynomial Transformation from $\mathcal{SHIQ}$ to $\mathcal{ALCI}b$

In this section, we present a stepwise satisfiability-preserving transformation from the quite common description logic $\mathcal{SHIQ}$ to the rather "exotic" $\mathcal{ALCI}b$. This will allow to apply the presented reasoning algorithm to terminological $\mathcal{SHIQ}$ knowledge bases.

5.1 From $\mathcal{SHIQ}$ to $\mathcal{ALCHIQ}$

As has been shown in [11], every $\mathcal{SHIQ}$ knowledge base KB can be converted into an equisatisfiable $\mathcal{ALCHIQ}$ knowledge base $\Theta_{\mathcal{S}}(\mathrm{KB})$, where $\mathcal{ALCHIQ}$ denotes the description logic $\mathcal{SHIQ}$ without transitivity axioms. Letting $\mathsf{clos}(\mathrm{KB})$ be the smallest set containing

- $\mathsf{NNF}(\neg C \sqcup D)$ for all $C \sqsubseteq D$ contained in the Tbox of KB,
- every subconcept of any concept contained in $\mathsf{clos}(\mathrm{KB})$,
- $\mathsf{NNF}(\neg C)$ for every $\leq n\, R.C \in \mathsf{clos}(\mathrm{KB})$, and
- $\forall S.C$ for every subrole[8] S of a role R where the Rbox of KB contains $\mathsf{Tra}(S)$ and where $\forall R.C \in \mathsf{clos}(\mathrm{KB})$,

this reduction is done as follows:

- remove all axioms $\mathsf{Tra}(R)$
- for every concept $\forall R.C$ from $\mathsf{clos}(\mathrm{KB})$ and every role S where $\mathsf{Tra}(S)$ is in KB and S is a subrole of R, add the axiom $\forall R.C \sqsubseteq \forall S.(\forall S.C)$.

Moreover, $\Theta_{\mathcal{S}}(\mathrm{KB})$ is polynomial in the size of KB.

[8] It is well known that determining whether a role is a subrole of another can be done by an easy syntactic Rbox check.

5.2 From $\mathcal{ALCHIQ}$ to $\mathcal{ALCHI}b^{\leq}$

Now we will show how any $\mathcal{ALCHIQ}$ knowledge base KB can be transformed into an $\mathcal{ALCHI}b^{\leq}$ knowledge base $\Theta_{\geq}(\mathrm{KB})$. In comparison to $\mathcal{ALCHIQ}$, $\mathcal{ALCHI}b^{\leq}$ disallows $\geq$ role restrictions but features restricted role expressions.
Given an $\mathcal{ALCHIQ}$ knowledge base KB, the $\mathcal{ALCHI}b^{\leq}$ knowledge base $\Theta_{\geq}(\mathrm{KB})$ is obtained by first flattening KB and then iteratively applying the following procedure to $\mathsf{FLAT}(\mathrm{KB})$ (terminating, if no qualified atleast number restrictions $\geq$ are left):

- Choose an occurrence of $\geq n\, R.A$ in the knowledge base.
- Substitute this occurrence by $\exists R_1.A \sqcap \ldots \sqcap \exists R_n.A$, where $R_1, \ldots, R_n$ are fresh role names.
- For every $i \in \{1, \ldots, n\}$, add $R_i \sqsubseteq R$ to the knowledge base's Rbox.
- For every $1 \leq i < k \leq n$, add $\forall(R_i \sqcap R_k).\bot$ to the knowledge base.

Observe that this transformation can be done in polynomial time.[9] It remains to show that KB and $\Theta_{\geq}(\mathrm{KB})$ are indeed equisatisfiable.

Lemma 3. *Let* KB *be an $\mathcal{ALCHIQ}$ knowledge base. Then the $\mathcal{ALCHI}b^{\leq}$ knowledge base $\Theta_{\geq}(\mathrm{KB})$ and* KB *are equisatisfiable.*

Proof. First we prove that every model of $\Theta_{\geq}(\mathrm{KB})$ is a model of KB. We do so by an inductive argument, showing that no additional models can be introduced by any substitution step of the above conversion procedure. Hence, assume KB$''$ is an intermediate knowledge base having a model $\mathcal{I}$ and KB$''$ is obtained from KB$'$ by eliminating the occurrence of $\geq n\, R.A$ as described above. Considering KB$''$, we find due to the KB$''$-axioms $\forall(R_i \sqcap R_k).\bot$ that no two individuals $\delta, \delta' \in \Delta^{\mathcal{I}}$ can be connected by more than one of the roles $R_1, \ldots, R_n$. In particular, this enforces $\delta' \neq \delta''$, whenever $\langle\delta, \delta'\rangle \in R_i^{\mathcal{I}}$ and $\langle\delta, \delta''\rangle \in R_j^{\mathcal{I}}$ for distinct R_i and R_j. Now consider an arbitrary δ from the extension of the concept $\exists R_1.A \sqcap \ldots \sqcap \exists R_n.A$. This assures the existence of individuals $\delta_1, \ldots, \delta_n$ with $\langle\delta, \delta_i\rangle \in R_i^{\mathcal{I}}$ and $\delta_i \in A^{\mathcal{I}}$ for $1 \leq i \leq n$. By the above observation, all those δ_i are pairwise distinct. Moreover, the axioms $R_i \sqsubseteq R$ ensure $\langle\delta, \delta_i\rangle \in R^{\mathcal{I}}$ for all i, hence we find that $\delta \in (\geq n\, R.A)^{\mathcal{I}}$. So we know $(\exists R_1.A \sqcap \ldots \sqcap \exists R_n.A)^{\mathcal{I}} \subseteq (\geq n\, R.C)^{\mathcal{I}}$. From the fact that both those concept expressions occur outside any negation or quantifier scope (as the transformation starts with a flattened knowledge base and does not itself introduce such nestings) in axioms $D'' \in \mathrm{KB}''$ and $D' \in \mathrm{KB}'$ which are equal up to the substituted occurrence, we can derive that $D''^{\mathcal{I}} \subseteq D'^{\mathcal{I}}$. Then, from $D''^{\mathcal{I}} = \Delta^{\mathcal{I}}$ follows $D'^{\mathcal{I}} = \Delta^{\mathcal{I}}$ making D' valid in $\mathcal{I}$. Apart from D', all other axioms from KB$'$ coincide with those from KB$''$ and hence are naturally satisfied in $\mathcal{I}$. So we find that $\mathcal{I}$ is a model of KB$'$.
At the end of our inductive chain, we finally arrive at $\mathsf{FLAT}(\mathrm{KB})$ which is equisatisfiable to KB by Proposition 1.
Second, we show that $\Theta_{\geq}(\mathrm{KB})$ has a model if KB has. Invoking Proposition 1 once more, satisfiability of KB entails the existence of a model of $\mathsf{FLAT}(\mathrm{KB})$. Moreover, every model of $\mathsf{FLAT}(\mathrm{KB})$ can be transformed to a model of $\Theta_{\geq}\mathrm{KB}$, as we will show

[9] Here we assume a unary encoding of the numbers n. Note that the same can be achieved for a binary encoding by using fresh roles as binary digits for complex roles, however, we stick to the easier presentation for the sake of understandability.

using the same inductive strategy as above by doing iterated model transformations following the syntactic knowledge base conversions. Again, assume KB$''$ is an intermediate knowledge base obtained from KB$'$ by eliminating the occurrence of $\geq n\,R.A$ as described above and suppose $\mathcal{I}$ is a model of KB$'$. Based on $\mathcal{I}$, we now (nondeterministically) construct an interpretation $\mathcal{J}$ as follows:

– $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$,
– for all $C \in \mathsf{N}_C$, let $C^{\mathcal{J}} := C^{\mathcal{I}}$,
– for all $S \in \mathsf{N}_R \setminus \{R_i \mid 1 \leq i \leq n\}$, let $S^{\mathcal{J}} := S^{\mathcal{I}}$,
– for every $\delta \in (\geq n\,R.A)^{\mathcal{I}}$, choose pairwise distinct $\epsilon_1^\delta, \ldots, \epsilon_n^\delta$ with $\langle\delta, \epsilon_i^\delta\rangle \in R^{\mathcal{I}}$ and $\epsilon_i^\delta \in A^{\mathcal{I}}$ (their existence being ensured by δ's abovementioned concept membership) and let $R_i^{\mathcal{J}} := \{\langle\delta, \epsilon_i^\delta\rangle \mid \delta \in (\geq n\,R.A)^{\mathcal{I}}\}$.

Now, it is easy to see that $\mathcal{J}$ satisfies all newly introduced axioms of the shape $\forall(R_i \sqcap R_k).\bot$, as the ϵ_i^δ have been chosen to be distinct for every δ. Moreover the axioms $R_i \sqsubseteq R$ are obviously satisfied by construction. Finally, for all $\delta \in (\geq n\,R.A)^{\mathcal{I}}$ the construction ensures $\delta \in (\exists R_1.A \sqcap \ldots \sqcap \exists R_n.A)^{\mathcal{J}}$ witnessed by the respective ϵ_i^δ. So we have $(\geq n\,R.A)^{\mathcal{I}} \subseteq (\exists R_1.A \sqcap \ldots \sqcap \exists R_n.A)^{\mathcal{J}}$. Now, again exploiting the fact that both those concept expressions occur in negation normalized universal concept axioms $D' \in$ KB$'$ and $D'' \in$ KB$''$ which are equal up to the substituted occurrence, we can derive that $D'^{\mathcal{I}} \subseteq D''^{\mathcal{J}}$. Then, from $D'^{\mathcal{I}} = \Delta^{\mathcal{I}}$ follows $D''^{\mathcal{J}} = \Delta^{\mathcal{J}}$ making D'' valid in $\mathcal{J}$. Apart from D' (and the newly introduced ones considered above), all other axioms from KB$''$ coincide with those from KB$'$ and hence are satisfied in $\mathcal{J}$, as they do not depend on the R_i whose interpretations are the only ones changed in $\mathcal{J}$ compared to $\mathcal{I}$. So we find that $\mathcal{J}$ is a model of KB$''$ □

5.3 From $\mathcal{ALCHI}b^{\leq}$ to $\mathcal{ALCI}b^{\leq}$

In the presence of restricted role expressions, role subsumption axioms can be easily transformed into Tbox axioms, as the subsequent lemma shows. This allows to dispense with role hierarchies in $\mathcal{ALCHI}b^{\leq}$ thereby restricting it to $\mathcal{ALCI}b^{\leq}$.

Lemma 4. *For any role names R, S, the Rbox axiom $R \sqsubseteq S$ and the Tbox axiom $\forall(R \sqcap \neg S).\bot$ are equivalent.*

Proof. By the semantics' definition, $R \sqsubseteq S$ holds in an interpretation $\mathcal{I}$ exactly if for every two individuals δ, δ' with $\langle\delta, \delta'\rangle \in R^{\mathcal{I}}$ also holds $\langle\delta, \delta'\rangle \in S^{\mathcal{I}}$. In turn, this is the case, if and only if there are no δ, δ' with $\langle\delta, \delta'\rangle \in R^{\mathcal{I}}$ but $\langle\delta, \delta'\rangle \notin S^{\mathcal{I}}$ (the last being expressible as $\langle\delta, \delta'\rangle \in (\neg S)^{\mathcal{I}}$). Furthermore, this condition can be formulated by $(R \sqcap \neg S)^{\mathcal{I}} = \emptyset$. Finally this is equivalent to $\forall(R \sqcap \neg S).\bot$. □

Hence, for any $\mathcal{ALCHI}b^{\leq}$ knowledge base KB, let $\Theta_{\mathcal{H}}$(KB) denote the $\mathcal{ALCI}b^{\leq}$ knowledge base obtained by substituting every Rbox axiom $R \sqsubseteq S$ by the Tbox axiom $\forall(R \sqcap \neg S).\bot$. The above lemma assures equivalence of KB and $\Theta_{\mathcal{H}}$(KB) (and hence also their equisatisfiability). Obviously, this reduction can be done in linear time.

5.4 From $\mathcal{ALCI}b^{\leq}$ to $\mathcal{ALCIF}b$

The elimination of the at-most concept descriptions $\leq$ from an $\mathcal{ALCI}b^{\leq}$ knowledge base is more intricate than the previously described transformations. Therefore, we subdivide it into two steps: first we eliminate concept expressions of the shape $\leq n\, R.C$ merely leaving axioms of the form $\leq 1\, R.\top$ (also known as role functionality statements) as the only occurrences of number restrictions, hence obtaining a $\mathcal{ALCIF}b$ knowledge base. Then, in a second step discussed in the next section, we eliminate all occurrences of axioms of the shape $\leq 1\, R.\top$.

So let KB an $\mathcal{ALCI}b^{\leq}$ knowledge base. We obtain the $\mathcal{ALCIF}b$ knowledge base $\Theta_{\mathcal{F}}(\mathrm{KB})$ by first flattening KB and then successively applying of the following steps (stopping when no more such occurrence is left):

- Choose an occurrence of the shape $\leq n\, R.A$ which is not a functionality axiom $\leq 1\, R.\top$,
- substitute this occurrence by $\forall(R \sqcap \neg R_1 \sqcap \ldots \sqcap \neg R_n).\neg A$ where $R_1, \ldots, R_n$ are fresh role names,
- for every $i \in \{1, \ldots, n\}$, add $\forall R_i.A$ as well as $\leq 1\, R_i.\top$ to the knowledge base.

Obviously, this transformation can be done in polynomial time, again assuming a unary encoding of the n. We now show that this conversion yields an equisatisfiable knowledge base. Structurally, the proof is very similar to that of Lemma 3.

Lemma 5. *Given an $\mathcal{ALCI}b^{\leq}$ knowledge base* KB*, the $\mathcal{ALCIF}b$ knowledge base* $\Theta_{\leq}(\mathrm{KB})$ *and* KB *are equisatisfiable.*

Proof. KB and $\mathsf{FLAT}(\mathrm{KB})$ are equisatisfiable by Proposition 1, so it remains to show equisatisfiability of $\mathsf{FLAT}(\mathrm{KB})$ and $\Theta_{\leq}(\mathrm{KB})$.

First, we prove that every model of $\Theta_{\leq}(\mathrm{KB})$ is a model of $\mathsf{FLAT}(\mathrm{KB})$. We do so in an inductive way by showing that no additional models can be introduced by any substitution step of the above conversion procedure. Hence, assume KB'' is an intermediate knowledge base having a model $\mathcal{I}$ and KB'' is obtained from KB' by eliminating the occurrence of $\leq n\, R.A$ as described above. Now consider an arbitrary δ from the extension of the concept $\forall(R \sqcap \neg R_1 \sqcap \ldots \sqcap \neg R_n).\neg A$. This ensures that whenever an individual $\delta' \in \Delta^{\mathcal{I}}$ satisfies $\langle \delta, \delta' \rangle \in R^{\mathcal{I}}$ and $\delta' \in A$, it must additionally satisfy $\langle \delta, \delta' \rangle \in R^{\mathcal{I}}$ for one $i \in \{1, \ldots, n\}$. However, it follows from the KB''-axioms $\leq 1\, R_i.\top$ that there is at most one such δ' for each R_i. Thus, there can be at most n individuals δ' with $\langle \delta, \delta' \rangle \in R^{\mathcal{I}}$ and $\delta' \in A$. This implies $\delta \in (\leq n\, R.A)^{\mathcal{I}}$. So we have $(\forall(R \sqcap \neg R_1 \sqcap \ldots \sqcap \neg R_n).\neg A)^{\mathcal{I}} \subseteq (\leq n\, R.A)^{\mathcal{I}}$. Due to the flattened knowledge base structure, both those concept expressions occur outside the scope of any negation or quantifier within axioms $D'' \in \mathrm{KB}''$ and $D' \in \mathrm{KB}'$ which are equal up to the substituted occurrence. Hence, we can derive that $D''^{\mathcal{I}} \subseteq D'^{\mathcal{I}}$. Then, from $D''^{\mathcal{I}} = \Delta^{\mathcal{I}}$ follows $D'^{\mathcal{I}} = \Delta^{\mathcal{I}}$ making D' valid in $\mathcal{I}$. Apart from D', all other axioms from KB' are contained in KB'' and hence are naturally satisfied in $\mathcal{I}$. So we find that $\mathcal{I}$ is a model of KB' as well.

Second, we show that every model of FLAT(KB) can be transformed to a model of $\Theta_{\leq}$(KB). We use the same induction strategy as above by doing iterated model transformations following the syntactic knowledge base conversions. Again, assume KB$''$ is an intermediate knowledge base obtained from KB$'$ by eliminating the occurrence of a $\leq n\,R.C$ as described above and suppose $\mathcal{I}$ is a model of KB$'$. Based on $\mathcal{I}$, we now (nondeterministically) construct an interpretation $\mathcal{J}$ as follows:

- $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$,
- for all $C \in \mathsf{N}_C$, let $C^{\mathcal{J}} := C^{\mathcal{I}}$,
- for all $S \in \mathsf{N}_R \setminus \{R_i \mid 1 \leq i \leq n\}$, let $S^{\mathcal{J}} := S^{\mathcal{I}}$,
- for every $\delta \in (\leq n\,R.A)^{\mathcal{I}}$, let $\epsilon_1^\delta, \ldots, \epsilon_k^\delta$ be an exhaustive enumeration (with arbitrary but fixed order) of all those $\epsilon \in \Delta^{\mathcal{I}}$ with $\langle\delta,\epsilon\rangle \in R^{\mathcal{I}}$ and $\epsilon \in A^{\mathcal{I}}$. Thereby δ's aforementioned concept membership ensures $k \leq n$. Now, let $R_i^{\mathcal{J}} := \{\langle\delta,\epsilon_i^\delta\rangle \mid \delta \in (\leq n\,R.A)^{\mathcal{I}}\}$.

Now, it is easy to see that $\mathcal{J}$ satisfies all newly introduced axioms of the shape $\leq 1\,R_i.\top$ as every δ has at most one R_i-successor (namely ϵ_i^δ, if $\delta \in (\leq n\,R.A)^{\mathcal{I}}$, and none otherwise). Moreover, the axioms $\forall R_i.A$ are satisfied, as the ϵ_i^δ have been chosen accordingly. Finally for all $\delta \in (\leq n\,R.A)^{\mathcal{I}}$ the construction ensures $\delta \in (\forall(R \sqcap \neg R_1 \sqcap \ldots \sqcap \neg R_n).\neg A)^{\mathcal{J}}$ as by construction, each R-successor of δ that lies within the extension of A is contained in $\epsilon_1^\delta, \ldots, \epsilon_k^\delta$ and therefore also R_i-successor of δ for some i. Now, again exploiting the fact that both those concept expressions occur in negation normalized universal concept axioms $D' \in$ KB$'$ and $D'' \in$ KB$''$ which are equal up to the substituted occurrence, we can derive that $D'^{\mathcal{I}} \subseteq D''^{\mathcal{J}}$. Then, from $D'^{\mathcal{I}} = \Delta^{\mathcal{I}}$ follows $D''^{\mathcal{J}} = \Delta^{\mathcal{J}}$ making D'' valid in $\mathcal{J}$. Apart from D'' (and the newly introduced ones considered above), all other axioms from KB$''$ coincide with those from KB$'$ and hence are satisfied in $\mathcal{J}$, as they do not depend on the R_i whose interpretations are the only ones changed in $\mathcal{J}$ compared to $\mathcal{I}$. So we find that $\mathcal{J}$ is a model of KB$''$ □

5.5 From $\mathcal{ALCIF}b$ to $\mathcal{ALCI}b$

In the sequel, we show how the role functionality axioms of the shape $\leq 1\,R.\top$ can be eliminated from an $\mathcal{ALCIF}b$ knowledge base while still preserving equisatisfiability. Essentially, we do so by adding axioms that enforce that for every functional role R, any two R-successors coincide with respect to their properties expressible in "relevant" DL terms. While it is rather obvious that those axioms follow from R's functionality, the other direction (a Leibniz-style "identitas indiscernibilium" argument) needs a closer look.

Taking an $\mathcal{ALCIF}b$ knowledge base KB, let $\Theta_{\mathcal{F}}$(KB) denote the $\mathcal{ALCI}b$ knowledge base obtained from KB by removing every role functionality axiom $\leq 1\,R.\top$ and instead adding

- $\forall R.\neg D \sqcup \forall R.D$ for every $D \in P(\text{KB} \setminus \{\leq 1\,R.\top \in \text{KB}\})$ as well as
- $\forall(R \sqcap S).\bot \sqcup \forall(R \sqcap \neg S).\bot$ for every atomic role S from KB.

Clearly, also this transformation can be done in polynomial time and space w.r.t. the size of KB.

Our goal is now to prove equisatisfiability of KB and $\Theta_{\mathcal{F}}(\mathrm{KB})$. The following lemma establishes the easier direction of this correspondency.

Lemma 6. *Any $\mathcal{ALCIF}b$ knowledge base* KB *entails all axioms of the $\mathcal{ALCI}b$ knowledge base $\Theta_{\mathcal{F}}(\mathrm{KB})$, i.e.* $\mathrm{KB} \models \Theta_{\mathcal{F}}(\mathrm{KB})$.

Proof. Let $\mathcal{J}$ be a model of KB. Obviously, $\mathcal{J}$ satisfies all axioms from $\mathrm{KB} \cap \Theta_{\mathcal{F}}(\mathrm{KB})$. It remains to consider the two kinds of axioms additionally introduced.
Firstly let D be an arbitrary concept. Now note that $\forall R.\neg D \sqcup \forall R.D$ is equivalent to the GCI $\exists R.D \sqsubseteq \forall R.D$. In words, this would mean that for any $\delta \in \Delta^{\mathcal{J}}$, all R-successors are in the extension of D whenever one of them is. Yet this is trivially satisfied if δ has at most one R-successor which is ensured by the corresponding functionality axiom in KB. Since we have shown the validity for arbitrary concepts D, this holds in particular for those from $P(\mathrm{KB} \setminus \{\leq 1\, R.\top \in \mathrm{KB}\})$.
Secondly, let S be an atomic role. Mark that $\forall(R \sqcap S).\bot \sqcup \forall(R \sqcap \neg S).\bot$ is equivalent to the GCI $\exists(R \sqcap S).\top \sqsubseteq \forall(R \sqcap \neg S).\bot$. This means that for any $\delta \in \Delta^{\mathcal{J}}$, all R-successors are also S-successors of it, whenever one of them is. Again, this is trivially satisfied as δ has at most one R-successor. □

The other direction for showing equisatisfiability, which amounts to finding a model of KB, given one for $\Theta_{\mathcal{F}}(\mathrm{KB})$, is somewhat more intricate and requires some intermediate considerations.

Lemma 7. *Let* KB *be an $\mathcal{ALCIF}b$ knowledge base and let* **F** *be the set of roles R with* $\leq 1\, R.\top \in \mathrm{KB}$.
Then in every model $\mathcal{J}$ of $\Theta_{\mathcal{F}}(\mathrm{KB})$, for every $\delta, \delta_1, \delta_2 \in \Delta^{\mathcal{J}}$ with $\langle \delta, \delta_1 \rangle \in R^{\mathcal{J}}$ and $\langle \delta, \delta_2 \rangle \in R^{\mathcal{J}}$, we have

- *for all $C \in P(\mathrm{KB} \setminus \{\leq 1\, R.\top \in \mathrm{KB}\})$, that $\delta_1 \in C^{\mathcal{J}}$ iff $\delta_2 \in C^{\mathcal{J}}$ as well as*
- *for all $S \in \mathsf{N}_R$, that $\langle \delta, \delta_1 \rangle \in S^{\mathcal{J}}$ iff $\langle \delta, \delta_2 \rangle \in S^{\mathcal{J}}$.*

Proof. For the first proposition, assume $\delta_1 \in C^{\mathcal{J}}$. From $\langle \delta, \delta_1 \rangle \in R^{\mathcal{J}}$ follows $\delta \in (\exists R.C)^{\mathcal{J}}$. Due to the $\Theta_{\mathcal{F}}(\mathrm{KB})$ axiom $\forall R.\neg C \sqcup \forall R.C$ (being equivalent to the GCI $\exists R.C \sqsubseteq \forall R.C$) follows $\delta \in (\forall R.C)^{\mathcal{J}}$. Since $\langle \delta, \delta_2 \rangle \in R^{\mathcal{J}}$, this implies $\delta_2 \in C^{\mathcal{J}}$. The other direction follows by symmetry.
To show the second proposition, assume $\langle \delta, \delta_1 \rangle \in S^{\mathcal{J}}$. Since also $\langle \delta, \delta_1 \rangle \in R^{\mathcal{J}}$, we have $\langle \delta, \delta_1 \rangle \in R \sqcap S^{\mathcal{J}}$ and hence $\delta \in (\exists(R \sqcap S).\top)^{\mathcal{J}}$. From the $\Theta_{\mathcal{F}}(\mathrm{KB})$ axiom $\forall(R \sqcap S).\bot \sqcup \forall(R \sqcap \neg S).\bot$ (which is equivalent to the GCI $\exists(R \sqcap S).\top \sqsubseteq \neg\exists(R \sqcap \neg S).\top$) we conclude $\delta \in (\neg\exists(R \sqcap \neg S).\top)^{\mathcal{J}}$, in words: δ has no R-successor that is not its S-successor. Thus, as $\langle \delta, \delta_2 \rangle \in R^{\mathcal{J}}$, it must also hold that $\langle \delta, \delta_2 \rangle \in S^{\mathcal{J}}$. Again, the other direction follows by symmetry.

□

In order to covert a model of $\Theta_{\mathcal{F}}(\mathrm{KB})$ into one of KB, one has to enforce role functionality where needed by cautiously deleting individuals from the original model without changing relevant concept memberships. The subsequent definition provides a method for this.

Definition 7. *Let $\mathcal{J}$ be an interpretation and let $\mathcal{I}$ be the domino interpretation of $\pi_C(\mathcal{J})$ of some concept set C. For a concept set $\mathcal{D} \subseteq C$, an interpretation $\mathcal{K}$ will be called* $\mathcal{D}$-pruning *of $\mathcal{I}$, if $\mathcal{K}$ can be constructed from $\mathcal{I}$ in the following way: set $\Delta_0 = \Delta^{\mathcal{I}}$ and then iteratively determine Δ_{i+1} from Δ_i as follows:*

- *Select a word-length minimal δ from Δ_i where there are distinct $\delta_1, \delta_2 \in \Delta_i$ with $\emptyset \neq \{R \in \mathsf{N}_R \mid \langle\delta, \delta_1\rangle \in R^{\mathcal{I}}\} = \{R \in \mathsf{N}_R \mid \langle\delta, \delta_2\rangle \in R^{\mathcal{I}}\}$ and $\{C \in P(\mathcal{D} \mid \delta_1 \in C^{\mathcal{I}}\} = \{C \in P(\mathcal{D} \mid \delta_2 \in C^{\mathcal{I}}\}$.*
- *Because of the construction of $\mathcal{I}$, for one of δ_1, δ_2 (w.l.o.g. say: δ_2) we have that $\delta_2 = \delta\langle\mathcal{A}, \mathcal{R}, \mathcal{B}\rangle$.*
  Delete δ_2 from Δ_i as well as all δ' having δ_2 as prefix.

Finally, let $\mathcal{K}$ be the limit of this process: $\Delta^{\mathcal{K}} := \bigcap_{i\in\mathbb{N}} \Delta_i$ and $\cdot^{\mathcal{K}}$ being the function $\cdot^{\mathcal{I}}$ restricted to $\Delta^{\mathcal{K}}$.

Roughly speaking, any $\mathcal{D}$*-pruning* of $\mathcal{I}$ is (nondeterministically) constructed by deleting successors not distinguishable w.r.t. the set of concept descriptions $\mathcal{D}$. Mark that the tree-like structure of the domino interpretation is crucial in order to make the process well-defined.

Lemma 8. *Let* KB *be an $\mathcal{ALCIF}b$ knowledge base, let $\mathcal{J}$ be a model of $\Theta_{\mathcal{F}}(\text{KB})$, and let $\text{KB}^* := \text{KB} \setminus \{\leq 1\, R.\top \in \text{KB}\}$. Then, any KB^*-pruning of $\mathcal{I}(\pi_{P(\Theta_{\mathcal{F}}(\text{KB}))}(\mathcal{J}))$ is a model of* KB.

Proof. By Proposition 2, $\mathcal{I} := \mathcal{I}(\pi_{P(\Theta_{\mathcal{F}}(\text{KB}))}(\mathcal{J}))$ is a model of $\Theta_{\mathcal{F}}(\text{KB})$, i.e., it fulfills all axioms from $\Theta_{\mathcal{F}}(\text{KB})$. Now let $\mathcal{K}$ be a KB^*-pruning of $\mathcal{I}$. For showing $\mathcal{K} \models \text{KB}$, we divide KB into two sets, namely the set of role functionality axioms KB^* and $\{\leq 1\, R.\top \in \text{KB}\}$ and show $\mathcal{K} \models \text{KB}^*$ and $\mathcal{K} \models \{\leq 1\, R.\top \in \text{KB}\}$ separately.

So, we start by showing $\mathcal{K} \models \text{KB}^*$.
We show this by proving that for each $C \in P(\text{KB}^*)$ and for every individual δ from $\mathcal{K}$, we have $\delta \in C^{\mathcal{K}}$ exactly if $\delta \in C^{\mathcal{I}}$. The claim for all Boolean combinations of elements from $P(\text{KB}^*)$ (and hence also the global validity of the axioms from KB^*) then follows by an easy structural induction.
We distinguish three cases (at places invoking the claim in an inductive way on formulae with smaller role depth):

- $C \in \mathsf{N}_C \cup \{\top, \bot\}$.
  Then the coincidence follows directly from the construction of $\mathcal{K}$.
- $C = \exists U.D$.
  "$\Rightarrow$"
  $\delta \in (\exists U.D)^{\mathcal{K}}$ means that there is a $\mathcal{K}$-individual δ' with $\langle\delta, \delta'\rangle \in U^{\mathcal{K}}$ and $\delta' \in D^{\mathcal{K}}$. Because of the construction of $\mathcal{K}$ by pruning $\mathcal{I}$, this means also $\langle\delta, \delta'\rangle \in U^{\mathcal{I}}$ and by induction hypothesis, we have $\delta' \in D^{\mathcal{I}}$, ergo $\delta \in (\exists U.D)^{\mathcal{I}}$.
  "$\Leftarrow$"
  If $\delta \in (\exists U.D)^{\mathcal{I}}$, there is an $\mathcal{I}$-individual δ' with $\langle\delta, \delta'\rangle \in U^{\mathcal{I}}$ and $\delta' \in D^{\mathcal{I}}$. In case δ' is not deleted during the construction of $\mathcal{K}$, it proves (by using the induction hypothesis on D) that $\delta \in (\exists U.D)^{\mathcal{K}}$. Otherwise, it must have been deleted due to

the existence of another $\mathcal{I}$-individual δ'' with $\{R \in \mathbf{R} \mid \langle\delta, \delta''\rangle \in R^{\mathcal{I}}\} = \{R \in \mathbf{R} \mid \langle\delta, \delta'\rangle \in R^{\mathcal{I}}\}$ and $\{E \in P(\mathrm{KB}^*) \mid \delta'' \in E^{\mathcal{I}}\} = \{E \in P(\mathrm{KB}^*) \mid \delta' \in E^{\mathcal{I}}\}$, which (w.l.o.g.) does not get deleted in the whole construction procedure. Yet, then the $\mathcal{K}$-individual δ'' obviously proves $\delta \in (\exists U.D)^{\mathcal{K}}$.

– $C = \forall R.D$.
  "⇒"
  Assume the contrary, i.e., $\delta \in (\forall U.D)^{\mathcal{K}}$ but $\delta \notin (\forall U.D)^{\mathcal{I}}$ which means that there is an $\mathcal{I}$-individual δ' with $\langle\delta, \delta'\rangle \in U^{\mathcal{I}}$ but $\delta' \notin D^{\mathcal{I}}$. In case δ' has not been deleted during the construction of $\mathcal{K}$, it disproves $\delta \in (\forall U.D)^{\mathcal{K}}$ (by invoking the induction hypothesis on D) leading to a contradiction. Otherwise, δ' is deleted because of the existence of another $\mathcal{I}$-individual δ'' with $\{R \in \mathbf{R} \mid \langle\delta, \delta''\rangle \in R^{\mathcal{I}}\} = \{R \in \mathbf{R} \mid \langle\delta, \delta'\rangle \in R^{\mathcal{I}}\}$ and $\{E \in P(\mathrm{KB}^*) \mid \delta'' \in E^{\mathcal{I}}\} = \{E \in P(\mathrm{KB}^*) \mid \delta' \in E^{\mathcal{I}}\}$, which (w.l.o.g.) does not get deleted in the whole construction procedure. Yet, then the $\mathcal{K}$-individual δ'' obviously contradicts $\delta \in (\exists U.D)^{\mathcal{K}}$.
  "⇐"
  Assume the contrary, i.e., $\delta \in (\forall U.D)^{\mathcal{I}}$ but $\delta \notin (\forall U.D)^{\mathcal{K}}$. The latter means that there is a $\mathcal{K}$-individual δ' with $\langle\delta, \delta'\rangle \in U^{\mathcal{K}}$ and $\delta' \notin D^{\mathcal{K}}$. Because of the construction of $\mathcal{K}$ by pruning $\mathcal{I}$, this means also $\langle\delta, \delta'\rangle \in U^{\mathcal{I}}$ and $\delta' \notin D^{\mathcal{I}}$, ergo $\delta \notin (\forall U.D)^{\mathcal{I}}$, contradicting the assumption.

We proceed by showing that every role R with $\leq 1\, R.\top \in \mathrm{KB}$ is functional in $\mathcal{K}$. Let $\delta \in \Delta^{\mathcal{K}}$. By Lemma 7 and the pointwise correspondence between $\mathcal{I}$ and $\mathcal{K}$ shown in the previous part of the proof, for any two R-successors δ_1, δ_2 of δ, two statements hold: Firstly, for all $C \in P(\mathrm{KB}^*)$, we have that $\delta_1 \in C^{\mathcal{K}}$ iff $\delta_2 \in C^{\mathcal{K}}$. Secondly, for all $S \in \mathsf{N}_R$ we have that $\langle\delta, \delta_1\rangle \in S^{\mathcal{K}}$ iff $\langle\delta, \delta_2\rangle \in S^{\mathcal{K}}$. However, in the pruning process generating $\mathcal{K}$, exactly such duplicate occurrences are erased, leaving at most one R-successor per δ. Thus we conclude $\delta_1 = \delta_2$.

So we end up having shown that all axioms from KB are satisfied in $\mathcal{K}$. □

Finally, we are ready to establish the equisatisfiability result also for this last transformation step.

Theorem 2. *For any $\mathcal{ALCIF}b$ knowledge base* KB*, the $\mathcal{ALCI}b$ knowledge base* $\Theta_{\mathcal{F}}(\mathrm{KB})$ *and* KB *are equisatisfiable.*

Proof. Lemma 6 ensures that every model of KB is also a model of $\Theta_{\mathcal{F}}(\mathrm{KB})$. Moreover, by Lemma 8, given a model $\mathcal{J}$ for of $\Theta_{\mathcal{F}}(\mathrm{KB})$, any KB^*-pruning of $\mathcal{I}(\pi_{P(\Theta_{\mathcal{F}}(\mathrm{KB}))}(\mathcal{J}))$ (obviously, the existence is assured by constructive definition) is a model of KB. This finishes the proof. □

In summary, we have shown in this section how to transform a $\mathcal{SHIQ}$ knowledge base KB into an equisatisfiable $\mathcal{ALCI}b$ knowledge base by calculating $\Theta_{\mathcal{F}}\Theta_{\leq}\Theta_{\mathcal{H}}\Theta_{\geq}\Theta_{\mathcal{S}}(\mathrm{KB})$. Moreover, as every of the single transformation steps is time polynomial, so is the overall procedure. Therefore, we are able to check the satisfiability of any $\mathcal{SHIQ}$ Tbox using the method presented in the preceding section, by first transforming it into $\mathcal{ALCI}b$ and then checking.

6 Related Work

The approach of constructing a canonical model (resp. a sufficient representation of it) in a downward manner (i.e. by pruning a larger structure) shows some similarity to Pratt's type elimination technique (see [12]), originally used to decide satisfiability of modal formulae.

Canonical models themselves have been a widely used notion in modal logic [13, 14], however, due to the additional expressive power of $\mathcal{ALCI}b$ compared to standard modal logics like K (being the modal logic counterpart of the description logic $\mathcal{ALC}$), we had to substantially modify the notion of a canonical model used there.

Very related in spirit (namely to use BDD-based reasoning for DL reasoning tasks and use a type elimination-like technique for doing so) is the work presented in [7]. However, the established results as well as the approaches differ greatly from ours: put into DL words, the authors establish a procedure for deciding the satisfiability of $\mathcal{ALC}$ concepts in a setting not allowing for general TBoxes, while our approach is able to check satisfiability of $\mathcal{SHIQ}$ (resp. $\mathcal{ALCI}b$) knowledge bases supporting general TBoxes, thereby generalizing the results from [7].

7 Conclusion and Outlook

The main contribution of this paper is that it provides a new algorithm for terminological reasoning in the description logic $\mathcal{SHIQ}$, based on ordered binary decision diagrams, which is a substantial improvement to [7]. Obviously, experiments will have to be done to investigate whether the conceptual insights – which indicate a competitive performance level – really work in practice. A prototype implementation is under way, and will be reported on in the future. OBDDs have shown excellent practical performance in structurally and computationally similar domains, so that some hope for practical applicability of this approach seem to be justified.

The major technical contributions in this paper are in fact two-fold.

To prove the correctness of our algorithm we had to elaborate on the model theoretic properties of $\mathcal{ALCI}b$. The technique was given in terms of Boolean functions being directly transferable into an algorithm based on OBDDs. We thereby provide the theoretical foundations for a novel paradigm for DL reasoning, which can be explored further not only in terms of implementations and evaluations, but also in other directions.

We also showed how a terminological $\mathcal{SHIQ}$ knowledge base can be converted into an equisatisfiable $\mathcal{ALCI}b$ knowledge base, thereby providing a foundational insight that reasoning in $\mathcal{SHIQ}$ can be done by developing reasoning solutions for $\mathcal{ALCI}b$. In particular, we showed that (qualified) number restrictions can be eliminated if allowing restricted complex role expressions.

Obviously, we intend to evaluate our approach by comparing it to well-established off-the-shelf reasoners, both tableau- and resolution-based approaches, and a prototype implementation is already under way. In fact, we are rather confident with respect to performance, as OBDDs have exhibited an excellent practical performance in structurally and computationally similar domains.

Besides implementation and evaluation, in the future we will extend our work towards Abox reasoning and to dealing with more expressive OWL DL constructs such as nominals.

References

1. McGuinness, D., v. Harmelen, F.: OWL Web Ontology Language Overview. W3C Recommendation (10 February 2004) Available at `http://www.w3.org/TR/owl-features/`.
2. Motik, B., Sattler, U.: A comparison of reasoning techniques for querying large description logic aboxes. In Hermann, M., Voronkov, A., eds.: Logic for Programming, Artificial Intelligence, and Reasoning, 13th International Conference, LPAR 2006, Phnom Penh, Cambodia, November 13-17, 2006, Proceedings. Volume 4246 of Lecture Notes in Computer Science., Springer (2006) 227–241
3. Bryant, R.E.: Graph-based algorithms for boolean function manipulation. IEEE Transactions on Computers **35** (1986) 677–691
4. Bryant, R.E.: Symbolic Boolean manipulation with ordered binary-decision diagrams. ACM Computing Surveys **24** (1992) 293–318
5. Huth, M.R.A., Ryan, M.D.: Logic in Computer Science: Modelling and reasoning about systems. Cambridge University Press (2000)
6. J.R. Burch, E.M. Clarke, K.L. McMillan, D.L. Dill, L.J. Hwang: Symbolic Model Checking: 10^{20} States and Beyond. In: Proceedings of the Fifth Annual IEEE Symposium on Logic in Computer Science, Washington, D.C., IEEE Computer Society Press (1990) 1–33
7. Pan, G., Sattler, U., Vardi, M.Y.: Bdd-based decision procedures for the modal logic k. Journal of Applied Non-Classical Logics **16** (2006) 169–208
8. Schild, K.: A correspondence theory for terminological logics: preliminary report. In: Proceedings of IJCAI-91, 12th International Joint Conference on Artificial Intelligence, Sidney, AU (1991) 466–471
9. Baader, F., Calvanese, D., McGuinness, D., Nardi, D., Patel-Schneider, P., eds.: The Description Logic Handbook: Theory, Implementation and Applications. Cambridge University Press (2007)
10. Tobies, S.: Complexity Results and Practical Algorithms for Logics in Knowledge Representation. PhD thesis, RWTH Aachen, Germany (2001)
11. Motik, B.: Reasoning in Description Logics using Resolution and Deductive Databases. PhD thesis, Universität Karlsruhe (TH), Germany (2006)
12. Pratt, V.R.: Models of program logics. In: Proc. 20th Annual Symposium on Foundations of Computer Science, San Juan, Puerto Rico (1979)
13. Popkorn, S.: First Steps in Modal Logic. Cambridge University Press (1994)
14. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press (2001)